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Factorization of $\mathcal{R}$-matrix and Baxter $\mathcal{Q}$-operators for generic $sl(N)$ spin chains

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Abstract
We develop an approach for constructing the Baxter $\mathcal{Q}$-operators for generic $sl(N)$ spin chains. The key element of our approach is the possibility of representing a solution of the Yang–Baxter equation in the factorized form. We prove that such a representation holds for a generic $sl(N)$ invariant $\mathcal{R}$-operator and find the explicit expression for the factorizing operators. Taking trace of monodromy matrices constructed of the factorizing operators one defines a family of commuting (Baxter) operators on the quantum space of the model. We show that a generic transfer matrix factorizes into the product of N Baxter $\mathcal{Q}$-operators and discuss an application of this representation for a derivation of functional relations for transfer matrices.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The notion of an $\mathcal{R}$-matrix plays a key role in the theory of lattice integrable systems. It is defined as a solution of the Yang–Baxter equation (YBE) [1]. The quantum inverse scattering method (QISM) [2–5] allows one to relate an exactly solvable model with each solution of the YBE and provides the methods of its analysis. These methods are the algebraic Bethe ansatz (ABA) [2, 6], the method of Baxter Q-operators [7] and separation of variables (SoV) [4]. The most widespread and well studied of them, ABA, depends crucially on the existence of a pseudovacuum state in the Hilbert space of a model. This requirement holds for a majority of the models which found applications in statistical mechanics and quantum field

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theory. Systems like the Toda chain [8] or noncompact spin magnets [9], for which the ABA method is not applicable, can be analyzed with the method of Baxter Q-operators and SoV. Unfortunately, these methods are less developed in comparison with ABA and their application was so far restricted to models with low rank symmetry groups. The main obstacle for the effective use of the method of Baxter Q-operators is the absence of a regular method of their construction. Nevertheless, in recent past there was certain progress in the field. The Baxter operators are now known for a number of models, see [9–31]. Special attention was paid to the analysis of the spin chain models with $sl(n)$ or $U_q(\widehat{sl}(n))$ symmetry and their supersymmetric extensions. In the approach developed by Bazhanov *et al* [11] the Baxter $\mathcal{Q}$-operators are constructed as a trace of the special monodromy matrix, the auxiliary space being an infinite-dimensional representation of the q-oscillator algebra. The method was applied to the models associated with the affine quantum algebras $U_q(\widehat{sl}(2))$ [11], $U_q(\widehat{sl}(3))$ [19] and $U_q(\widehat{sl}(2|1))$ [31]. The construction relies on the explicit solution for the universal $\mathcal{R}$-matrix [32–34]. Since the latter is rather complicated, it causes technical problems if one attempts to extend the analysis to a general, $U_q(\widehat{sl}(N))$, case [30].

The spin chain models associated with $sl(N=2,3)$ and $sl(2|1)$ algebras were analyzed in [28, 35–37] by a different method. It results in much the same functional relations among the Baxter $\mathcal{Q}$-operators and transfer matrices as for the q-deformed models. This method relies heavily upon the special representation for the $\mathcal{R}$-operator. Namely, it was shown in [38] that the $sl(N=2,3)$ and $sl(2|1)$ invariant $\mathcal{R}$ matrices on a tensor product of generic representations can be represented as a product of two ($sl(2)$) and three ($sl(3)$ and $sl(2|1)$) simpler operators. These operators possess a number of remarkable properties which are easily translated into the properties of the corresponding transfer matrices. There is no doubt that the factorized representation for $\mathcal{R}$-operator exists for a general N. The factorizing operators were constructed in the explicit form in [29] for an invariant $\mathcal{R}$-operator acting on the tensor product of the principal continuous series representations of $SL(N,\mathbb{C})$ group. The aim of the present paper is to show that the factorization holds for an invariant $\mathcal{R}$-operator on the tensor product of generic (infinite-dimensional) highest weight representations of the $sl(N)$ algebra (Verma modules).

The Verma module over $sl(N)$ algebra can be realized as a vector space of polynomials of $N(N-1)/2$ variables of an arbitrary degree, see section 3. We will use this realization throughout the paper. The defining equations for the factorizing operators on Verma modules are too complicated to be solved directly for a general N. To find a solution we will use the results of [29] where the invariant $\mathcal{R}$ operator acting on the tensor product of the principal series representation of $SL(N,\mathbb{C})$ group was constructed. This construction uses the properties of the intertwining operators for the principal series representations and naturally gives rise to the factorized form of the $\mathcal{R}$-operator. The building blocks for the $\mathcal{R}$-operator are simple integral operators defined on functions from $L^2(Z\times Z)$, where Z is the group of lower unitriangular matrices. We try to interpret these operators as operators on Verma modules. Below we show that such an interpretation is possible and find explicit expressions for the factorizing operators.

As an illustration let us briefly consider the simplest case of the $SL(2,\mathbb{C})$ group. The $SL(2,\mathbb{C})$ invariant $\mathcal{R}$-operator acts on the tensor product of two unitary principal series representations, $T^{(s_1,\bar{s}_1)}\otimes T^{(s_2,\bar{s}_2)}$,

$$[T^{(s,\bar{s})}(g)f](z) = (cz+d)^{-2s}(\bar{c}\bar{z}+\bar{d})^{-2\bar{s}} f\left(\frac{az+b}{cz+d}\right),$$

$\bar{s}=1-s^*$, $2(s-\bar{s})=n$, $f\in L^2(\mathbb{C})$. It can be represented in the factorized form [29]

$$\mathcal{R}(u-v) = V(u_1-u_2)S(v_1-u_1)V(v_1-u_2)U(u_1-v_2)S(v_2-u_2)U(u_2-u_1). \tag{1.1}$$

All operators U, V, S depend on two spectral parameters, holomorphic and antiholomorphic ones, the latter (antiholomorphic) is not displayed explicitly, i.e. $U(\lambda) \equiv U(\lambda, \bar\lambda)$, with $\lambda - \bar\lambda = n$ being integer. In equation (1.1) we have put $u_1 = u + 1 - s_1, u_2 = u + s_1, v_1 = v + 1 - s_2, v_2 = v + s_2$, and similarly for the barred parameters, $\bar u_1 = \bar u + 1 - \bar s_1$, etc. The operators U and V are integral operators

$$
\begin{aligned}
[U(\lambda) f](z, w) &= A(\lambda) \int \mathrm{d}^2\xi\, \xi^{-1-\lambda}\bar\xi^{-1-\bar\lambda} f(z - \xi, w), \\
[V(\lambda) f](z, w) &= A(\lambda) \int \mathrm{d}^2\eta\, \eta^{-1-\lambda}\bar\eta^{-1-\bar\lambda} f(z, w - \eta)
\end{aligned}
\tag{1.2}
$$

(the factor $A(\lambda)$ is defined in equation (2.36)), while the operator $S(\lambda)$ is a multiplication operator

$$
[S(\lambda) f](z, w) = (z - w)^{-\lambda} (\bar z - \bar w)^{-\bar\lambda} f(z, w). \tag{1.3}
$$

For $\lambda^* = -\bar\lambda$ all operators are unitary operators with respect to the standard scalar product on $L^2(\mathbb{C} \times \mathbb{C})$. It is clear that an interpretation of the operators U, V and S as operators on the product of Verma modules causes a lot of problems and hardly possible at all. However, for the products of U, V and S operators, $R^{(1)} = VSV$ and $R^{(2)} = USU$, such interpretation exists. Indeed, the action of the operator $R^{(2)}$ on a test function can be represented in the form[4]

$$
[R^{(2)} f](z, w) = A(u_2 - v_2) \int \mathrm{d}^2\xi [1 - \xi]^{v_2 - u_2 - 1} [\xi]^{u_1 - v_2} f(\xi(z - w) + w, w), \tag{1.4}
$$

where we put for brevity $[\xi]^\lambda \equiv \xi^\lambda \bar\xi^{\bar\lambda}$. Let us explore this integral for the case when $f(z, w)$ is a holomorphic polynomial in z and w. It is clear that the result is a polynomial again provided that the integrals

$$
I_k(\alpha, \beta) = \int \mathrm{d}^2\xi\, \xi^k [1 - \xi]^{-\alpha} [\xi]^{-\beta}, \tag{1.5}
$$

where $\alpha = 1 + u_2 - v_2$ and $\beta = v_2 - u_1$, converge for an arbitrary k in some region of the parameters α and β. Let us assume that $\bar\alpha - \alpha = n > 0$ and $\bar\beta - \beta = m > 0$. The integral (1.5) converges in the vicinity of the singular points $\xi = 0, 1, \infty$ if $\mathrm{Re}\,\alpha < 1, \mathrm{Re}\,\beta < 1$ and $\mathrm{Re}\,\bar\alpha + \bar\beta > 1$, respectively. These conditions can always be satisfied. Thus equation (1.4) defines an operator on the product of Verma modules. It could be checked that this operator coincides with the factorizing operator obtained in [38]. Next, the integral (1.5) is an analytic function of α, β,

$$
I_k(\alpha, \beta) = \pi(-1)^{n+m+k} \frac{\Gamma(n - \alpha)}{\Gamma(\alpha)} \frac{\Gamma(m - \beta)}{\Gamma(\beta - k)} \frac{\Gamma(n + m + \alpha + \beta - 1)}{\Gamma(2 + k - \alpha - \beta)},
$$

hence one can extend the domain of definition of operator (1.4) to arbitrary α, β. Thus we have constructed the factorizing operator on the tensor product of $sl(2)$ Verma modules starting from the solution for the $SL(2, \mathbb{C})$ case.

In what follows we extend these arguments to a general case of $sl(N)$ invariant $\mathcal{R}$ operator. Instead of a direct calculation of integrals (which becomes too cumbersome) we accept another approach. An operator on the Verma module is determined by its matrix in some basis. However, such a description is not very convenient. It is preferable to describe an operator by its kernel which is defined as follows, $A(z, w) = \sum_{kn} e_k(z) A_{kn} h_n(w)$, where $e_k(z)$ and $h_n(w)$ are basis vectors in the Verma module and its dual space. The function

[4] The derivation of equation (1.4) makes use of the so-called star–triangle relation [39], see e.g. [9, 29] for details.

$A(z, w)$ depends on the functional realization of the dual space, but as the latter is fixed there is a one-to-one correspondence between an operator and its kernel. Proceeding in this way we derive the defining equation for the kernel of the factorizing operator. Instead of solving this equation we show that the 'kernel' of $SL(N, \mathbb{C})$ operator calculated in some specific basis satisfies the same equation and, hence, provides the kernel of the factorizing operator on the Verma module. The factorizing operators in the $SL(N, \mathbb{C})$ case possess a number of remarkable properties. The heuristic arguments given above imply that the operators obtained by '$SL(N, \mathbb{C}) \to sl(N)$ reduction' should inherit all these properties.

Taking the trace of monodromy matrices constructed from the factorizing operators one defines a family of commuting operators (Baxter Q-operators) on the quantum space of the model. The trace is taken over an infinite-dimensional space and, in general, diverges for the model with unbroken $sl(N)$ symmetry. The finiteness of the traces can be provided by introducing a regulating factor [15, 19, 24, 25] which breaks $sl(N)$ symmetry down to its diagonal subgroup. We prove that in this case the traces for the Baxter operators converge absolutely. Moreover, the Baxter operators can be identified with the generic $sl(N)$ transfer matrix with a specially chosen auxiliary space. Using the properties of the factorizing operators we show that a generic transfer matrix can be represented as a product of Baxter Q-operators. This representation is quite helpful for the study of the functional relations among the transfer matrices. Though we will mainly discuss the homogeneous spin chains the approach is applicable to the analysis of the inhomogeneous spin chain models.

The paper is organized as follows: in section 2, we recall some basic facts about representations of $SL(N, \mathbb{C})$ group and fix the notation. We give also the summary of the results of [29] which will be necessary for further discussion. In section 3, the $sl(N)$ invariant $\mathcal{R}$-operator on the tensor product of two Verma modules is constructed. To handle some technical problems we introduce an invariant bilinear form on a Verma module and describe an operator by its kernel with respect to this form. Using this technique we find the explicit form of the factorizing operators in the $sl(N)$ case. We prove that these operators obey certain commutation relations. In section 4, we construct the Baxter operators and study their properties. A generic invariant transfer matrix is defined as the trace of the monodromy matrix over an infinite-dimensional auxiliary space. To ensure convergence of the trace we introduce a boundary operator which breaks the $sl(N)$ symmetry of the transfer matrix to its diagonal subalgebra. We prove that the corresponding trace over an infinite-dimensional space exists and show that a generic transfer matrix factorizes into a product of N Baxter Q-operators. Section 5 contains concluding remarks. The derivation of some technically involved results is presented in the appendices.

2. $SL(N, \mathbb{C})$ invariant $\mathcal{R}$-operator: principal series representations

To make further discussion self-contained we give here a brief review of the principal series representations of the $SL(N, \mathbb{C})$ group, and then formulate the results of [29] which will be used in subsequent analysis.

2.1. Principal series representation of $SL(N, \mathbb{C})$

The unitary principal series representations of the group $SL(N, \mathbb{C})$ can be realized on the space of functions on the group of lower triangular $N \times N$ matrices [40, 41]. Namely, let $Z_-(Z_+)$ and $H_+(H_-)$ be the groups of lower (upper) unitriangular matrices and upper (lower) triangular matrices with unit determinant, respectively,

$$z = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ z_{21} & 1 & 0 & \dots & 0 \\ z_{31} & z_{32} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ z_{N1} & z_{N2} & \dots & z_{N,N-1} & 1 \end{pmatrix} \in Z_-, \qquad h = \begin{pmatrix} h_{11} & h_{12} & h_{13} & \dots & h_{1N} \\ 0 & h_{22} & h_{23} & \dots & h_{2N} \\ 0 & 0 & h_{33} & \dots & h_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & h_{N,N} \end{pmatrix} \in H_+.$$

Almost any matrix $g \in G = SL(N, \mathbb{C})$ admits the Gauss decomposition $g = zh$. The element $z_1 \in Z_-$ satisfying the condition $g^{-1} \cdot z = z_1 \cdot h$ will be denoted by $z\bar{g}$, so that $g^{-1}z = z\bar{g} \cdot h$. The homomorphism $g \to T^\alpha(g)$,

$$[T^\alpha(g)\Phi](z) = \boldsymbol{\alpha}(h^{-1})\Phi(z\bar{g}), \tag{2.1}$$

defines a principal series representation of the group $SL(N, \mathbb{C})$ on a suitable space of functions on the group Z_-, $\Phi(z) = \Phi(z_{21}, \bar{z}_{21}, z_{31}, \bar{z}_{31}, \dots, z_{NN-1}, \bar{z}_{NN-1})$ [40, 41]. The function $\boldsymbol{\alpha}$ in equation (2.1) is the character of the group H_+,

$$\boldsymbol{\alpha}(h) = \prod_{k=1}^{N} h_{kk}^{-\sigma_k - k} \bar{h}_{kk}^{-\bar{\sigma}_k - k}. \tag{2.2}$$

Here $\bar{h}_{kk} \equiv (h_{kk})^*$ is the complex conjugate of h_{kk}, whereas in general $\sigma_k^* \neq \bar{\sigma}_k$. We put $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_N)$ and will sometimes use notation $T^{\boldsymbol{\sigma}}$ instead of T^α. Since $\det h = 1$ the function $\boldsymbol{\alpha}(h)$ depends only on the differences $\sigma_{k,k+1} \equiv \sigma_k - \sigma_{k+1}$ and can be rewritten in the form

$$\boldsymbol{\alpha}(h) = \prod_{k=1}^{N-1} (\Delta_k(h))^{1-\sigma_{k,k+1}} (\bar{\Delta}_k(h))^{1-\bar{\sigma}_{k,k+1}} = \prod_{k=1}^{N-1} (\Delta_k(h))^{n_k} |\Delta_k(h)|^{2(1-\bar{\sigma}_{k,k+1})}, \tag{2.3}$$

where $n_k = \bar{\sigma}_{k,k+1} - \sigma_{k,k+1}$, $k = 0, \dots, N-1$ are integer numbers, $n_k \in \mathbb{Z}$.[5] The function $\Delta_k(M)$ is defined by

$$\Delta_k(M) = \det M_k, \tag{2.4}$$

where the $k \times k$ matrix M_k is the kth main minor of the matrix M, and $\bar{\Delta}_k(M) = (\Delta_k(M))^*$. That is $\Delta_k(h) = \prod_{i=1}^{k} h_{ii}$ for $h \in H_+$. We will assume that the parameters σ_k satisfy the restriction

$$\sigma_1 + \sigma_2 + \dots + \sigma_N = N(N-1)/2. \tag{2.5}$$

The operator $T^\alpha(g)$ is a unitary operator on the Hilbert space $L^2(Z_-)$,

$$\langle \Phi_1 | \Phi_2 \rangle = \int Dz (\Phi_1(z))^* \Phi_2(z), \qquad Dz = \prod_{1 \leqslant i < k \leqslant N} \mathrm{d}^2 z_{ki},$$

if the character $\boldsymbol{\alpha}'(h) = \boldsymbol{\alpha}(h) \prod_{k=1}^{N} |h_{kk}|^{2k}$ is a unitary one, i.e. $|\boldsymbol{\alpha}'| = 1$. This condition holds if $\sigma_{k,k+1}^* + \bar{\sigma}_{k,k+1} = 0$ for $k = 1, \dots, N-1$, i.e.

$$\sigma_{k,k+1} = -\frac{n_k}{2} + \mathrm{i}\lambda_k, \qquad \bar{\sigma}_{k,k+1} = \frac{n_k}{2} + \mathrm{i}\lambda_k, \qquad k = 1, 2, \dots, N-1, \tag{2.6}$$

where n_k is an integer and λ_k is real. The unitary principal series representation T^α is irreducible. Two representations T^α and $T^{\alpha'}$ are unitary equivalent if and only if the corresponding parameters $(\sigma_1, \dots, \sigma_N)$ and $(\sigma'_1, \dots, \sigma'_N)$ are related by a permutation, see for details [40, 41].

[5] From now on, since each variable a comes along with its antiholomorphic twin $\bar{a}$ we will write down only the holomorphic variant of equations.

We will also need the explicit expression for the generators of infinitesimal $SL(N,\mathbb{C})$ transformations, which are defined in the standard way

$$\left[T^{\alpha}\left(\mathbb{1}+\sum\nolimits_{ik}\epsilon^{ik}\mathcal{E}_{ki}\right)\Phi\right](z)=\Phi(z)+\sum\nolimits_{ik}(\epsilon^{ik}E_{ki}+\bar{\epsilon}^{ik}\bar{E}_{ki})\Phi(z)+O(\epsilon^2).$$

Here $\mathcal{E}_{ik}$ $(1\leqslant i,k\leqslant N)$, are the generators in the fundamental representation of the $SL(N,\mathbb{C})$ group,

$$(\mathcal{E}_{ik})_{nm}=\delta_{in}\delta_{km}-\frac{1}{N}\delta_{ik}\delta_{nm}. \tag{2.7}$$

The generators $E_{ik}(\bar{E}_{ik})$ are linear differential operators in the variables $z_{mn}(\bar{z}_{mn})$, $(1\leqslant n<m\leqslant N)$ which satisfy the commutation relation

$$[E_{ik},E_{nm}]=\delta_{kn}E_{im}-\delta_{im}E_{nk}. \tag{2.8}$$

The generators E_{ik} admit the following representation [29]:

$$E_{ik}=-\sum_{m\leqslant n}z_{km}(D_{nm}+\delta_{nm}\sigma_m)(z^{-1})_{ni}. \tag{2.9}$$

Here D_{nm}, $n>m$ are the generators of the right shifts, $\Phi(z)\to\Phi(zz_0)$,

$$\Phi\left(z\left(\mathbb{1}+\sum\nolimits_{k>i}\epsilon^{ik}\mathcal{E}_{ki}\right)\right)=\left(1+\sum\nolimits_{k>i}(\epsilon^{ik}D_{ki}+\bar{\epsilon}^{ik}\bar{D}_{ki})+O(\epsilon^2)\right)\Phi(z). \tag{2.10}$$

Equation (2.9) can be written in the matrix form

$$E=-z\,(\sigma+D)\,z^{-1}, \tag{2.11}$$

where

$$E=\sum_{ik}E_{ik}e_{ki},,\qquad D=\sum_{i>k}D_{ik}e_{ki},\qquad \sigma=\sum_{k}\sigma_k e_{kk} \tag{2.12}$$

and the matrices e_{nm} $(n,m=1,\ldots,N)$ form the standard basis in the space $\mathrm{Mat}(N\times N)$, $(e_{nm})_{ik}=\delta_{in}\delta_{mk}$. The generators D_{ki} satisfy the same commutation relation as E_{ki}, (equation 2.8) $[D_{ki},D_{nm}]=\delta_{in}D_{km}-\delta_{km}D_{ni}$ and commute with the generators of left shifts, E_{ki}, $k>i$. An explicit expression for the generators of left and right shifts reads

$$E_{ki}=-\sum_{m=1}^{i}z_{im}\frac{\partial}{\partial z_{km}}=\sum_{m=k}^{N}\tilde{z}_{mk}\frac{\partial}{\partial\tilde{z}_{mi}}, \tag{2.13}$$

$$D_{ki}=-\sum_{m=1}^{i}\tilde{z}_{im}\frac{\partial}{\partial\tilde{z}_{km}}=\sum_{m=k}^{N}z_{mk}\frac{\partial}{\partial z_{mi}}, \tag{2.14}$$

where $\tilde{z}_{ki}=(z^{-1})_{ki}$ and we recall that $z_{ii}=1$. Let us note here that the operator D_{ki} depends on the variables in the kth and ith columns of the matrix z, or on the variables in the kth and ith rows of the inverse matrix z^{-1}.

2.2. Coherent states

In this subsection we describe the system of functions with 'good' transformation properties with respect to $SL(N,\mathbb{C})$ transformations. This system will play a key role in establishing relationship between $\mathcal{R}$-operators defined on different spaces of functions. Namely, we define

$$\Delta^{\sigma}(z,\alpha)=\prod_{k=1}^{N-1}(\Delta_k(\alpha z))^{-1+\sigma_{k,k+1}}(\bar{\Delta}_k(\alpha z))^{-1+\bar{\sigma}_{k,k+1}}, \tag{2.15}$$

where $\Delta_k(M) = \det M_k$, see equation (2.4). The function $\Delta^\sigma(z, g^{-1})$ is nothing but the prefactor $\boldsymbol{\alpha}$ in equation (2.1), i.e. Δ^σ is a transformation of unity,

$$\Delta^\sigma(z, g^{-1}) = T^\alpha(g) \cdot 1 = \boldsymbol{\alpha}(h^{-1}(z, g)). \tag{2.16}$$

We will refer to the function $\Delta^\sigma(z, \alpha)$, with α being an upper triangular matrix, $\alpha \in Z_+$, as a coherent state. For a unitary representation the system of coherent states $\{\Delta^\sigma(z, \alpha), \alpha \in Z_+\}$ is a complete orthogonal system in $L^2(Z)$. Indeed, it is easy to verify that the integral operator $\mathfrak{D}$ defined as

$$[\mathfrak{D}\varphi](z) = \int D\alpha\, \Delta^\sigma(z, \alpha)\varphi(\alpha), \quad \alpha \in Z_+, \qquad D\alpha = \prod_{1 \leqslant i < k \leqslant N} \mathrm{d}^2\alpha_{ik} \tag{2.17}$$

intertwines the unitary representations T^α and $\widetilde{T}^\gamma$,

$$T^\alpha(g)\mathfrak{D} = \mathfrak{D}\widetilde{T}^\gamma(g). \tag{2.18}$$

The representation $\widetilde{T}^\gamma$ is defined on functions on the group Z_+,

$$[\widetilde{T}^\gamma(g) f](\alpha) = \gamma(h) f(\alpha\bar{g}), \qquad \gamma(h) = \prod_{k=1}^{N-1} (\Delta_k(h))^{-1-\sigma_{k,k+1}} (\bar{\Delta}_k(h))^{-1-\bar{\sigma}_{k,k+1}}. \tag{2.19}$$

Here for $\alpha \in Z_+$ and $g \in SL(N, \mathbb{C})$ we put $\alpha \cdot g = h \cdot \alpha\bar{g}, h \in H_-$. It follows from equation (2.18) that the operator $\mathfrak{D}\mathfrak{D}^\dagger$ commutes with all operators $T^\alpha(g)$. Since $T^\alpha(g)$ is an operator-irreducible representation [40] the operator $\mathfrak{D}\mathfrak{D}^\dagger$ is proportional to the unit operator. Hence

$$\int Dz\, \Delta^\sigma(z, \alpha)\overline{\Delta^\sigma(z, \alpha')} = c_N(\boldsymbol{\sigma}) \prod_{1 \leqslant k < i \leqslant N} \delta^2(\alpha_{ik} - \alpha'_{ik}), \tag{2.20}$$

where $\delta^2(z) = \delta(x)\delta(y)$ for complex $z = x + \mathrm{i}y$. For the normalization factor we obtained

$$c_N(\boldsymbol{\sigma}) = \prod_{1 \leqslant k < i \leqslant N} \frac{\pi^2}{|\sigma_i - \sigma_k|^2}.$$

As was mentioned above coherent states have good transformation properties. Namely, one easily derives

$$\boldsymbol{\alpha}^{-1}(h_+(z, g))\Delta^\sigma(z\bar{g}, \alpha) = \boldsymbol{\alpha}^{-1}(h_-(\alpha, g^{-1}))\Delta^\sigma(z, \alpha\bar{g}^{-1}), \tag{2.21}$$

where, we recall, $g^{-1} \cdot z = z\bar{g} \cdot h_+(z, g), \alpha \cdot g^{-1} = h_-(\alpha, g^{-1}) \cdot \alpha\bar{g}^{-1}$. Equation (2.21) can be brought into the form

$$T_z^\sigma(g)\Delta^\sigma(z, \alpha) = \widetilde{T}_\alpha^{-\sigma}(g^{-1})\Delta^\sigma(z, \alpha), \tag{2.22}$$

where the transformation $\widetilde{T}^{-\sigma}(g)$ is given by equations (2.19) with the substitution $\sigma_k \to -\sigma_k$. Thus the coherent state $\Delta^\sigma(z, \alpha)$ satisfies the equation

$$\left(E_{ik}^{(z)} + \widetilde{E}_{ik}^{(\alpha)}\right)\Delta^\sigma(z, \alpha) = 0, \tag{2.23}$$

where E_{ik} and $\widetilde{E}_{ik}, i, k = 1, \ldots, N$, are the generators of the $sl(N)$ algebra in the representations T^σ and $\widetilde{T}^{-\sigma}$, respectively.

2.3. Factorized form of $\mathcal{R}$ operator

The Yang–Baxter equation is an operator equation which is a corner stone of the theory of integrable systems. It has the form

$$\mathcal{R}_{12}(u-v)\mathcal{R}_{13}(u-w)\mathcal{R}_{23}(v-w) = \mathcal{R}_{23}(v-w)\mathcal{R}_{13}(u-w)\mathcal{R}_{12}(u-v). \tag{2.24}$$

The operators act on the tensor product $\mathbb{V}_1\otimes\mathbb{V}_2\otimes\mathbb{V}_3$. As usual, an operator with the indices ik acts nontrivially on the space $\mathbb{V}_i\otimes\mathbb{V}_k$ only, (i.e. $\mathcal{R}_{12} = R_{12}\otimes\mathbb{I}_3$ and so on). We are interested in $SL(N,\mathbb{C})$ invariant solutions of the YBE. Therefore, it will be assumed that each space carries a certain representation of the $SL(N,\mathbb{C})$ group. For a special choice, $\mathbb{V}_3 = \mathbb{V}_f$, where $\mathbb{V}_f$ is the fundamental representation of $SL(N,\mathbb{C})$, the YBE turns into the RLL relation. It has the form

$$\mathcal{R}_{12}(u-v)L_1(u)L_2(v) = L_2(v)L_1(u)\mathcal{R}_{12}(u-v), \tag{2.25}$$

where $L(u)$ is the Lax operator

$$L(u) = u + \sum_{mn} e_{mn}E_{nm}. \tag{2.26}$$

With the help of equation (2.9) the Lax operator (2.26) can be represented as (see equation (2.11))

$$L(u) = z(u-\sigma-D)z^{-1}. \tag{2.27}$$

The form of the operators of right shifts does not depend on a representation, hence the Lax operator is completely determined by the set of numbers (spectral parameters) $\{u_1,\ldots,u_N\}$, where $u_k = u-\sigma_k$, i.e. $L(u) = L(u_1,\ldots,u_N)$.

A solution of the YBE for the principal series representations of the $SL(N,\mathbb{C})$ group, i.e. an $\mathcal{R}$ operator which acts on the tensor product of the principal series representations $T^\alpha\otimes T^\beta$,[6] has the factorized form [29]

$$\mathcal{R}_{12}(u-v) = P_{12}\mathbb{R}^{(1)}_{12}(u_1-v_1)\mathbb{R}^{(2)}_{12}(u_2-v_2)\cdots\mathbb{R}^{(N)}_{12}(u_N-v_N). \tag{2.28}$$

Here P_{12} is the permutation operator, $P_{12}f(z,w) = f(w,z)$ and the parameters u_k, v_k are defined as follows, $u_k = u-\sigma_k$ and $v_k = v-\rho_k$. All operators depend on holomorphic and antiholomorphic spectral parameters, i.e. $\mathbb{R}^{(k)}(\lambda) = \mathbb{R}^{(k)}(\lambda,\bar\lambda)$, which are subjected to the restriction $\lambda-\bar\lambda\in\mathbb{Z}$, but for brevity we display the holomorphic variables only.

The defining equation for the operator $\mathbb{R}^{(k)}$ is

$$\begin{aligned}\mathbb{R}^{(m)}_{12}&(u_m-v_m)L_1(u_1,\ldots,u_m,\ldots u_N)L_2(v_1,\ldots,v_m,\ldots,v_N)\\ &= L_1(u_1,\ldots,v_m,\ldots,u_N)L_2(v_1,\ldots,u_m,\ldots,v_N)\mathbb{R}^{(m)}_{12}(u_m-v_m)\end{aligned} \tag{2.29}$$

and similar for the antiholomorphic Lax operators. Thus the operator $\mathbb{R}^{(m)}_{12}(u_m-v_m)$ exchanges the spectral parameters u_m and v_m in the Lax operators. It follows from equation (2.29) that $\mathbb{R}^{(m)}(\lambda)$ intertwines the representations $T^\alpha\otimes T^\beta$ and $T^{\alpha_{m,\lambda}}\otimes T^{\beta_{m,-\lambda}}$,

$$\mathbb{R}^{(m)}(\lambda)T^\alpha(g)\otimes T^\beta(g) = T^{\alpha_{m,\lambda}}(g)\otimes T^{\beta_{m,-\lambda}}(g)\mathbb{R}^{(m)}(\lambda), \tag{2.30}$$

where

$$\boldsymbol{\alpha}_{m,\lambda}(h) = h^{-\lambda}_{mm}\bar h^{-\bar\lambda}_{mm}\boldsymbol{\alpha}(h), \qquad \boldsymbol{\beta}_{m,-\lambda}(h) = h^{\lambda}_{mm}\bar h^{\bar\lambda}_{mm}\boldsymbol{\beta}(h).$$

The operator $\mathbb{R}^{(m)}$ is completely determined by the 'quantum numbers' of the representations they act on, i.e. by the characters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, ($\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$) and the spectral parameter λ. We will

[6] We accept the following 'standard' parametrization for the characters, $\alpha(h) = \prod_{k=1}^N h_{kk}^{-\sigma_k-k}\bar h_{kk}^{-\bar\sigma_k-k}$, $\beta(h) = \prod_{k=1}^N h_{kk}^{-\rho_k-k}\bar h_{kk}^{-\bar\rho_k-k}$. Also, we will use the letters z and w for arguments of the functions from the representation space of $T^\alpha\otimes T^\beta$, $f(z,w)$.

display the spectral parameter λ as an argument of the operator, $\mathbb{R}^{(m)}(\lambda|\boldsymbol{\alpha},\boldsymbol{\beta}) \to \mathbb{R}^{(m)}(\lambda)$, and omit the dependence on $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, assuming that these parameters are always fixed by a representation the operator acts on.

The solution to equation (2.29) can be represented in the form [29]

$$\mathbb{R}^{(m)}_{12}(u_m - v_m) = \left(\overleftarrow{\prod_{i=1}^{m-1}} \mathbb{U}_i(u_i - v_m)\right)\left(\overrightarrow{\prod_{j=m}^{N-1}} \mathbb{V}_j(u_m - v_{j+1})\right) \\ \times \mathbb{S}(u_m - v_m)\left(\overleftarrow{\prod_{j=m}^{N-1}} \mathbb{V}_j(v_{j+1} - v_m)\right)\left(\overrightarrow{\prod_{i=1}^{m-1}} \mathbb{U}_i(u_m - u_i)\right). \tag{2.31}$$

The operator $\mathbb{S}(\lambda)$ is a multiplication operator

$$\mathbb{S}(\lambda) f(z,w) = [(w^{-1}z)_{N1}]^{\lambda} f(z,w), \tag{2.32}$$

where $[a]^{\lambda} \equiv a^{\lambda}\bar{a}^{\bar{\lambda}}$. The function $[a]^{\lambda}$ is single valued only if $\lambda - \bar{\lambda} \in Z$. This condition is always satisfied in the above construction. The ordered products are defined as follows:

$$\overrightarrow{\prod_{i=1}^{m}} A_i = A_1 A_2 \cdots A_m \qquad \text{and} \qquad \overleftarrow{\prod_{i=1}^{m}} A_i = A_m A_{m-1} \cdots A_1.$$

The operators $\mathbb{U}_i(\lambda) \equiv \mathbb{U}_i(\lambda,\bar{\lambda})$ and $\mathbb{V}_i(\lambda) \equiv \mathbb{V}_i(\lambda,\bar{\lambda})$ are unitary operators if $\lambda^* + \bar{\lambda} = 0$. They can be expressed in terms of the operators of right shifts, equation (2.13), acting on z and w variables, respectively

$$\mathbb{U}_i(\lambda,\bar{\lambda}) = \left(D^{(z)}_{i+1,i}\right)^{\lambda}\left(\bar{D}^{(z)}_{i+1,i}\right)^{\bar{\lambda}}, \tag{2.33}$$

$$\mathbb{V}_j(\lambda,\bar{\lambda}) = \left(D^{(w)}_{j+1,j}\right)^{\lambda}\left(\bar{D}^{(w)}_{j+1,j}\right)^{\bar{\lambda}}. \tag{2.34}$$

The operators (2.33) are well defined and can be represented as integral operators [29],

$$[\mathbb{U}_i(\lambda)\Phi](z) = A(\lambda)\int d^2\zeta\,[\zeta]^{-1-\lambda}\Phi(z_\zeta), \tag{2.35}$$

where $[\zeta]^{\sigma} = \zeta^{\sigma}\bar{\zeta}^{\bar{\sigma}}$, $z_\zeta = z(1 - \zeta e_{i+1,i})$ and

$$A(\lambda) \stackrel{\text{def}}{=} A(\lambda,\bar{\lambda}) = \frac{1}{\pi} i^{\bar{\lambda}-\lambda}\Gamma(1+\lambda)/\Gamma(-\bar{\lambda}). \tag{2.36}$$

The operator $\mathbb{U}_k$, with the spectral parameter $\lambda = \sigma_{k,k+1} = \sigma_k - \sigma_{k+1}$, intertwines the representations, T^{α} and $T^{\alpha'}$,

$$\mathbb{U}_k(\sigma_{k,k+1}) T^{\alpha}(g) = T^{\alpha'}(g)\mathbb{U}_k(\sigma_{k,k+1}), \tag{2.37}$$

where $\boldsymbol{\alpha}'(h) = (h_{kk}/h_{k+1,k+1})^{\sigma_{k,k+1}}\boldsymbol{\alpha}(h)$.

2.4. *Properties of factorizing operators*

The operators $\mathbb{R}^{(k)}$ satisfy a number of remarkable relations [29]

$$\mathbb{R}^{(m)}_{12}(0) = \mathbb{1}, \tag{2.38a}$$

$$\mathbb{R}^{(m)}_{12}(\lambda)\mathbb{R}^{(m)}_{12}(\mu) = \mathbb{R}^{(m)}_{12}(\lambda+\mu), \tag{2.38b}$$

$$\mathbb{R}^{(m)}_{12}(\lambda)\mathbb{R}^{(n)}_{23}(\mu) = \mathbb{R}^{(n)}_{23}(\mu)\mathbb{R}^{(m)}_{12}(\lambda), \qquad (n \neq m), \tag{2.38c}$$

$$\mathbb{R}^{(m)}_{12}(\lambda)\mathbb{R}^{(m)}_{23}(\lambda+\mu)\mathbb{R}^{(m)}_{12}(\mu) = \mathbb{R}^{(m)}_{23}(\mu)\mathbb{R}^{(m)}_{12}(\lambda+\mu)\mathbb{R}^{(m)}_{23}(\lambda), \tag{2.38d}$$

$$\mathbb{R}^{(m)}_{12}(\lambda-\sigma_m+\rho_m)\mathbb{R}^{(n)}_{12}(\lambda-\sigma_n+\rho_n) = \mathbb{R}^{(n)}_{12}(\lambda-\sigma_n+\rho_n)\mathbb{R}^{(m)}_{12}(\lambda-\sigma_m+\rho_m). \tag{2.38e}$$

These relations are sufficient to prove that the $\mathcal{R}$-operator (2.28) satisfies the YBE [29]. Next, taking into account equations (2.13), (2.33) and (2.31) one deduces the following commutation relations for the operators $\mathbb{R}^{(m)}$:

$$\mathbb{R}^{(m)}(\lambda)(z^{-1})_{kj} = (z^{-1})_{kj}\mathbb{R}^{(m)}(\lambda), \qquad \text{for} \qquad k > m, \tag{2.39a}$$

$$\mathbb{R}^{(m)}(\lambda)w_{kj} = w_{kj}\mathbb{R}^{(m)}(\lambda), \qquad \text{for} \qquad j < m \tag{2.39b}$$

and

$$D^{(z)}_{k+1,k}\mathbb{R}^{(m)}(\lambda) = \mathbb{R}^{(m)}(\lambda)D^{(z)}_{k+1,k}, \qquad \text{for} \qquad k > m+1, \tag{2.40a}$$

$$D^{(w)}_{k+1,k}\mathbb{R}^{(m)}(\lambda) = \mathbb{R}^{(m)}(\lambda)D^{(w)}_{k+1,k}, \qquad \text{for} \qquad k < m-1. \tag{2.40b}$$

Finally, one can see from the representation (2.31) that the operator $\mathbb{R}^{(m)}$ depends only on the spectral parameters, $u_1, \ldots, u_m$ and $v_m, \ldots, v_N$. Namely, it depends on the spectral parameter $\lambda = u_m - v_m$ and

$$u_i - u_m = \sigma_m - \sigma_i, \qquad i < m, \qquad \text{and} \qquad v_m - v_j = \rho_j - \rho_m, \qquad j > m. \tag{2.41}$$

It means that

$$\mathbb{R}^{(m)}(\lambda|\boldsymbol{\alpha},\boldsymbol{\beta}) = \mathbb{R}^{(m)}(\lambda|\boldsymbol{\alpha}',\boldsymbol{\beta}') \tag{2.42}$$

if the characters satisfy the following relations:

$$\frac{\boldsymbol{\alpha}(h)}{\boldsymbol{\alpha}'(h)} = f(\Delta_m(h),\ldots,\Delta_{N-1}(h)) \qquad \text{and} \qquad \frac{\boldsymbol{\beta}(h)}{\boldsymbol{\beta}'(h)} = \varphi(\Delta_1(h),\ldots,\Delta_{m-1}(h)).$$

That is the ratio $\boldsymbol{\alpha}(h)/\boldsymbol{\alpha}'(h)$ does not depend on Δ_k, $k = 1,\ldots,m-1$ and similarly for $\boldsymbol{\beta}$.

2.5. Factorizing operators in the coherent states basis

Let us calculate the action of the operator $\mathbb{R}^{(m)}$ in the coherent state basis

$$\Delta^{\sigma\rho}(z,w|\alpha,\beta) \equiv \Delta^{\sigma}(z,\alpha)\Delta^{\rho}(w,\beta), \tag{2.43}$$

where $z, w \in Z_-$ and $\alpha, \beta \in Z_+$. We will use the following notations:

$$\alpha z = z_\alpha d_{z,\alpha}\alpha_z, \qquad \beta w = w_\beta d_{w,\beta}\beta_w, \tag{2.44}$$

where $d_{z,\alpha}, d_{w,\beta}$ are diagonal matrices, $z_\alpha, w_\beta \in Z_-$ and $\alpha_z, \beta_w \in Z_+$.

Lemma 1. *The action of the operator* $\mathbb{R}^{(m)}_{12}(\lambda)$ *on the state (2.43) is given by*

$$\left[\mathbb{R}^{(m)}_{12}(\lambda)\Delta^{\sigma\rho}\right](z,w|\alpha,\beta) = f^{(m)}_{\sigma\rho}(\lambda)\left[\left(\beta_w w^{-1}z\alpha_z^{-1}\right)_{mm}\right]^{\lambda}\Delta^{\sigma\rho}(z,w|\alpha,\beta), \tag{2.45}$$

where $[a]^\lambda \equiv a^\lambda\bar{a}^{\bar\lambda}$. *The prefactor* $f^{(m)}_{\sigma\rho}(\lambda)$ *is*

$$\begin{aligned} f^{(m)}_{\sigma\rho}(\lambda) &= \prod_{k=1}^{m-1}(-1)^{\lambda-\bar\lambda}\frac{A(\lambda-\sigma_{km})}{A(-\sigma_{km})}\prod_{j=m+1}^{N}\frac{A(\lambda-\rho_{mj})}{A(-\rho_{mj})} \\ &= \prod_{k=1}^{m-1}(-1)^{\lambda-\bar\lambda}\frac{A(u_k-v_m)}{A(u_k-u_m)}\prod_{j=m+1}^{N}\frac{A(u_m-v_j)}{A(v_m-v_j)}, \end{aligned} \tag{2.46}$$

where

$$A(\lambda) = \frac{1}{\pi} i^{\bar{\lambda}-\lambda}\Gamma(1+\lambda)/\Gamma(-\bar{\lambda})$$

and

$$\lambda = u_m - v_m, \qquad u_m = u - \sigma_k, \qquad v_k = v - \rho_k. \tag{2.47}$$

The proof of the lemma is rather technical and can be found in appendix A. Now we want to discuss equation (2.45) in more detail. First, we note that the rhs of equation (2.45) is given (up to the prefactor $f^{(m)}_{\sigma\rho}(\lambda)$) by the product of two functions

$$\mathcal{K}^m_{\lambda,\sigma,\rho}(z,w|\alpha,\beta) = \left(\beta_w w^{-1} z \alpha_z^{-1}\right)^{\lambda}_{mm} \left(\prod_{k=1}^{N-1} (\Delta_k(\alpha z))^{\sigma_{k,k+1}-1} (\Delta_k(\beta w))^{\rho_{k,k+1}-1}\right) \tag{2.48}$$

and

$$\bar{\mathcal{K}}^m_{\bar{\lambda},\bar{\sigma},\bar{\rho}}(z,w|\alpha,\beta) = \left(\left(\beta_w w^{-1} z \alpha_z^{-1}\right)^*_{mm}\right)^{\bar{\lambda}} \left(\prod_{k=1}^{N-1} (\Delta_k(z^\dagger\alpha^\dagger))^{\bar{\sigma}_{k,k+1}-1} (\Delta_k(w^\dagger\beta^\dagger))^{\bar{\rho}_{k,k+1}-1}\right). \tag{2.49}$$

The function $\mathcal{K}(\bar{\mathcal{K}})$ is an (anti)holomorphic function of z, w, α, β in the vicinity of the point $z = w = \alpha = \beta = 1$.

Further, it follows from equation (2.29) that the function $\mathcal{K}^m_{\lambda,\sigma,\rho}(z,w|\alpha,\beta)$ satisfies the following equation:

$$\begin{aligned}\widetilde{L}^{\alpha}_1(u_1,\ldots,u_m,\ldots u_N)\widetilde{L}^{\beta}_2(v_1,\ldots,v_m,\ldots,v_N)\mathcal{K}^m_{u_m-v_m,\sigma,\rho}(z,w|\alpha,\beta)\\ = L^z_1(u_1,\ldots,v_m,\ldots,u_N)L^w_2(v_1,\ldots,u_m,\ldots,v_N)\mathcal{K}^m_{u_m-v_m,\sigma,\rho}(z,w|\alpha,\beta).\end{aligned} \tag{2.50}$$

Here the Lax operators $\widetilde{L}_1(\widetilde{L}_2)$ are given by

$$\widetilde{L}(u) = u - \sum_{mn} e_{mn}\widetilde{E}_{nm}, \tag{2.51}$$

where the generators $\widetilde{E}_{nm}$ correspond to the representation $\widetilde{T}^{-\sigma}(\widetilde{T}^{-\rho})$, see equations (2.22), (2.19). The superscript of the Lax operator (L^z, $\widetilde{L}^{\alpha}$) indicates the variable it acts on. Equation (2.29) follows directly from equations (2.23), (2.29) and (2.45). In the following section, we show that the function $\mathcal{K}^m_{\lambda,\sigma,\rho}(z,w|\alpha,\beta)$ defines a factorizing operator on the tensor product of Verma modules.

3. $sl(N)$ invariant $\mathcal{R}$ operator for generic highest weight representations

In this section we construct $sl(N)$ invariant solution of the YBE on the tensor product of two generic highest weight representations of the $sl(N)$ Lie algebra (Verma modules).

Let $\mathbb{V}$ be a linear space of polynomials of arbitrary degree in z_{ki},

$$\mathbb{V} = \{P(z_{21}, z_{31}, \ldots, z_{NN-1}), \deg(P) < \infty\}. \tag{3.1}$$

The homomorphism

$$\pi^\sigma : \mathcal{E}_{ki} \to E_{ki} = -\sum_{m\leqslant n} z_{im}(D_{nm} + \delta_{nm}\sigma_m)(z^{-1})_{nk} \tag{3.2}$$

(see equation (2.12) for a definition of D_{nm}) defines a representation of the $sl(N)$ algebra on the space $\mathbb{V}$. The operators E_{ki} are completely determined by the parameters $\{\sigma_1,\ldots,\sigma_N\}$.

More precisely, they depend on the differences $\sigma_{nm} = \sigma_n - \sigma_m$. In order to stress a similarity with the $SL(N,\mathbb{C})$ case, it is convenient to specify the representation by a function $\boldsymbol{\alpha}(h) = \prod_{k=1}^{N} h_{kk}^{-k-\sigma_k}$, where h is an upper triangular matrix with unit determinant. Henceforth, we will use both notations, π^{α} and π^{σ}, for a representation of the $sl(N)$ algebra. A representation π^{σ} is irreducible if none of the differences $\sigma_{ik} = \sigma_i - \sigma_k$, $i < k$, is a positive integer [42, 43]. We will assume that this condition is fulfilled.

The highest weight vector, υ_0, ($E_{ik}\upsilon_0 = 0$, for $i > k$) of the Verma module (3.1) is given by $\upsilon_0 = 1$. For the highest weight $\boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_{N-1})$, $((E_{kk} - E_{k+1,k+1})\cdot\upsilon_0 \equiv \lambda_k\upsilon_0)$ one finds $\lambda_k = \sigma_{k+1} - \sigma_k + 1$. In case if all components of the weight vector $\boldsymbol{\lambda}$ are negative integer, $\lambda_k = -n_k, n_k \geqslant 0, k = 1,\ldots,N-1$ then the Verma module has a finite-dimensional invariant subspace. This subspace is a finite dimensional representation of $sl(N)$ algebra which corresponds to the Young tableau specified by the partition $\{\ell_1,\ldots,\ell_{N-1}\}$, where $\ell_k = \sum_{i=k}^{N-1} n_k$ is the length of the kth row in the Young tableau.

3.1. Bilinear form and kernel of an operator

We define the following linear combinations of the Cartan generators E_{kk} and the unit operator,

$$H_p = \sum_{k=1}^{p}(E_{kk} + \sigma_k + k - N) = \sum_{m=p+1}^{N}\sum_{k=1}^{p} z_{mk}\frac{\partial}{\partial z_{mk}} = \sum_{m=p+1}^{N}\sum_{k=1}^{p} \tilde{z}_{mk}\frac{\partial}{\partial \tilde{z}_{mk}}, \tag{3.3}$$

where $p = 1,\ldots,N-1$ and $\tilde{z} = z^{-1}$. The space $\mathbb{V}$ is a direct sum of the weight subspaces $\mathbb{V}_h$,

$$\mathbb{V} = \sum_{h\in\mathbb{Z}_+^{N-1}} \oplus\mathbb{V}_h, \qquad \mathbb{V}_h = \{v\in\mathbb{V}|(H_p - h_p)v = 0, p = 1,\ldots,N-1\}. \tag{3.4}$$

Each subspace $\mathbb{V}_h$ has a finite dimension. The union of the bases in all $\mathbb{V}_h$ gives a basis in $\mathbb{V}$. We will mostly use the following basis:

$$e_n(z) = \prod_{i>k} z_{ik}^{n_{ik}}, \tag{3.5}$$

where n is a multi-index, $n = \{n_{21},\ldots,n_{NN-1}\}$.

Let $\overline{\mathbb{V}}$ be a linear space of polynomials of arbitrary degree in $\bar{z}_{ki} = z^*_{ki}$,

$$\overline{\mathbb{V}} = \{P(\bar{z}_{21},\bar{z}_{31},\ldots,\bar{z}_{NN-1}), \deg(P) < \infty\} \tag{3.6}$$

and φ be an antilinear map $\mathbb{V}\to\overline{\mathbb{V}}$ defined by $\varphi(e_n) = \bar{e}_n = \prod_{i>k}\bar{z}_{ik}^{n_{ik}}$. We put $\overline{\mathbb{V}}_h = \varphi(\mathbb{V}_h)$.

Let Ω be a bilinear form on the product $\overline{\mathbb{V}}\times\mathbb{V}$ such that

$$\Omega(\bar{v},u) = 0, \qquad \text{if} \qquad \bar{v}\in\overline{\mathbb{V}}_h \qquad \text{and} \qquad u\in\mathbb{V}_{h'}, \qquad h\neq h' \tag{3.7a}$$

$$\text{the matrix} \qquad \Omega_{nm} = \Omega(\bar{e}_n, e_m) \qquad \text{is invertible.} \tag{3.7b}$$

Let $\mathbb{A}$ be a linear operator on the space $\mathbb{V}$ and A_{nm} be its matrix in the basis e_n, $\mathbb{A}e_n = \sum_m e_m A_{mn}$. We will refer to a function

$$\mathcal{A}(z,w) = \sum_{nm} e_n(z)(A\Omega^{-1})_{nm}\overline{e_m(w)} \tag{3.8}$$

as a kernel of the operator $\mathbb{A}$. The kernel $\mathcal{A}(z,w)$ is an (anti)holomorphic function in $z(w)$ in the vicinity of the point $z = w = 1$ ($z_{ik} = w_{ik} = 0$) on condition that the series converges. An action of the operator $\mathbb{A}$ on an arbitrary vector from $\mathbb{V}$ (which is a polynomial in z) can be represented as

$$[\mathbb{A}P](z) = \Omega(\mathcal{A}(z,w), P(w)). \tag{3.9}$$

The bilinear form Ω is completely determined by a kernel of the unit operator (reproducing kernel)

$$\mathcal{I}(z,w)=\sum_{nm}e_n(z)\Omega^{-1}_{nm}\overline{e_m(w)}. \tag{3.10}$$

It is clear that the kernel of an operator does not depend on the choice of the basis. In particular, the kernel $A(z,w)$ of the operator $\mathbb{A}$ can be obtained as

$$\mathcal{A}(z,w)=\mathbb{A}\mathcal{I}(z,w). \tag{3.11}$$

Let π^σ be an irreducible representation of the $sl(N)$ algebra on the vector space $\mathbb{V}$. Henceforth we will assume that the space $\mathbb{V}$ is equipped with a bilinear form Ω_σ such that the reproducing kernel has the form

$$\mathcal{I}^\sigma(z,w)=\prod_{k=1}^{N-1}(\Delta_k(w^\dagger z))^{\sigma_{k,k+1}-1}. \tag{3.12}$$

It is easy to derive from equations (2.1) and (2.19) that the kernel $\mathcal{I}^\sigma(z,w)$ satisfies the equation

$$E^{(z)}_{ki}\mathcal{I}^\sigma(z,w)=-\widetilde{E}^{(\bar w)}_{ki}\mathcal{I}^\sigma(z,w), \tag{3.13}$$

where $E_{ik}=\pi^\sigma(e_{ik})$ and $\widetilde{E}_{ki}=\tilde\pi^{-\sigma}(e_{ik})$ are the holomorphic generators corresponding to the representations T^σ (equation (2.1)) and $\widetilde{T}^{-\sigma}$ (equations(2.22) and (2.19)).[7] It follows from equation (3.13) that the form Ω_σ (for the irreducible representation) satisfies the properties (3.7). From equation (3.13) one derives

$$\Omega_\sigma(\widetilde{E}_{ki}Q,P)+\Omega_\sigma(Q,E_{ki}P)=0. \tag{3.14}$$

As usual, a bilinear form on the tensor product of two (or more) representations $\pi^\sigma\otimes\pi^\rho$, is defined as

$$\Omega_{\sigma\rho}(Q_1\otimes Q_2,P_1\otimes P_2)=\Omega_\sigma(Q_1,P_1)\cdot\Omega_\rho(Q_2,P_2).$$

It will be useful to have a more functional definition for the bilinear form Ω_σ. Let us put for any two polynomials $P(z)$ and $Q(\bar z)$

$$B_\sigma(Q,P)=c_N(\boldsymbol{\sigma})\int Dz\,\mu_\sigma(z)Q(\bar z)P(z), \tag{3.15}$$

where

$$\mu_\sigma(z)=\prod_{k=1}^{N-1}(\Delta_k(z^\dagger z))^{-\sigma_{k,k+1}-1}. \tag{3.16}$$

Since P and Q are polynomials, the integral in (3.15) converges in some region of $\sigma_{k,k+1}$ and, as will be shown later, defines a meromorphic function of σ_{ik}, which we take for a definition of the lhs of (3.15) for arbitrary σ_{ik}. Note that for positive integer $\sigma_{k,k+1}$ the integral (3.15) defines the invariant $SU(N)$ scalar product. It is easy to check that the bilinear form (3.15) with the measure (3.16) results in equation (3.14). Hence the bilinear form B_σ, equation (3.15) and the form Ω_σ coincide up to a prefactor. The two forms coincide identically at the following normalization:

$$c_N(\boldsymbol{\sigma})=\pi^{-N(N-1)/2}\left(\prod_{1\leqslant i<k\leqslant N}\sigma_{ik}\right) \tag{3.17}$$

[7] The representation $\widetilde{T}^{-\sigma}$ was defined on functions of the group Z_+, $f(\alpha)$, so that $\widetilde{E}^{(\bar w)}_{ki}=\widetilde{E}_{ki}(\alpha)|_{\alpha=w^\dagger}$.

$(B_\sigma(Q,P)=1=\Omega_\sigma(Q,P)$ for $Q(\bar z)=P(z)=1)$. For example, for $N=2$ equation (3.15) becomes

$$\Omega_\sigma(Q,P)=\frac{2j+1}{\pi}\int \mathrm{d}^2z\,\frac{Q(\bar z)P(z)}{(1+z\bar z)^{2j+2}}, \tag{3.18}$$

where $2j=\sigma_{12}-1$.

It is useful to extend the space $\mathbb{V}$ in order to include into consideration non-polynomial functions which are analytic in the vicinity of the point $z=1(z_{ik}=0)$. This allows us to consider finite transformations of functions from $\mathbb{V}$, $P(z)\to f(z)=\alpha(h)P(z\bar g)$. The function $f(z)$ is no more a polynomial, however, if g is sufficiently close to unity then $f(z)$ is an analytic function of z in the vicinity of the point $z=1$ $(f(z)=\sum_n c_n e_n(z)=\sum_h f_h(z)$, where f_h is a projection of the vector f to the weight subspace $\mathbb{V}_h)$. For such functions the bilinear form is defined as the sum, $\Omega_\sigma(\bar\psi,f)=\sum_h\Omega_\sigma(\bar\psi_h,f_h)$, in case if the series converges.

3.2. $\mathcal{R}$-matrix and factorizing operators

Our immediate purpose in this subsection is to construct the factorizing operators $\mathbb{R}^{(m)}$, $m=1,\dots,N$ which act on the tensor product of two Verma modules $\mathbb{V}\otimes\mathbb{V}$ and solve the RLL relation

$$\mathbb{R}^{(m)}_{12}(u_m-v_m)L_1(u_1,\dots,u_m,\dots u_N)L_2(v_1,\dots,v_m,\dots,v_N)$$
$$=L_1(u_1,\dots,v_m,\dots,u_N)L_2(v_1,\dots,u_m,\dots,v_N)\mathbb{R}^{(m)}_{12}(u_m-v_m). \tag{3.19}$$

The parameters u_m,v_m are defined by equation (2.47). It is straightforward to check that the operator $\mathbb{R}^{(m)}_{12}(\lambda)$ intertwines the representations $\pi^\alpha\otimes\pi^\beta$ $(\equiv\pi^\sigma\otimes\pi^\rho)$ and $\pi^{\alpha_{m,\lambda}}\otimes\pi^{\beta_{m,-\lambda}}$ $(\equiv\pi^{\sigma'}\otimes\pi^{\rho'})$,

$$\mathbb{R}^{(m)}_{12}(\lambda)\pi^\alpha\otimes\pi^\beta=\pi^{\alpha_{m,\lambda}}\otimes\pi^{\beta_{m,-\lambda}}\mathbb{R}^{(m)}_{12}(\lambda), \tag{3.20}$$

where $\alpha_{m,\lambda}(h)=h_{mm}^{-\lambda}\alpha(h)$ and $\beta_{m,-\lambda}(h)=h_{mm}^{\lambda}\beta(h)$.

Let $\mathcal{R}^{(m)}_\lambda(z,w|\bar\alpha,\bar\beta)$ be a kernel of the operator $\mathbb{R}^{(m)}_{12}$,

$$\left[\mathbb{R}^{(m)}_{12}\psi\right](z,w)=\Omega_{\sigma'\rho'}\big(\mathcal{R}^{(m)}_\lambda(z,w|\bar\alpha,\bar\beta),\psi(\alpha,\beta)\big).$$

Using property (3.14) one easily derives the defining equation for $\mathcal{R}^{(m)}_\lambda(z,w|\bar\alpha,\bar\beta)$

$$\widetilde L_1^{\bar\alpha}(u_1,\dots,u_m,\dots u_N)\widetilde L_2^{\bar\beta}(v_1,\dots,v_m,\dots,v_N)\mathcal{R}^{(m)}_\lambda(z,w|\bar\alpha,\bar\beta)$$
$$=L_1^z(u_1,\dots,v_m,\dots,u_N)L_2^w(v_1,\dots,u_m,\dots,v_N)\mathcal{R}^{(m)}_\lambda(z,w|\bar\alpha,\bar\beta), \tag{3.21}$$

where $\lambda=u_m-v_m$. We recall here that

$$\widetilde L_1(u_i)=u-\sum_{nm}e_{nm}\widetilde E^{(1)}_{mn}$$

and similarly for $\widetilde L_2$. Let us note that equation (3.21) coincides identically with equation (2.50) whose solution is given by the function $\mathcal{K}^m_{\lambda,\sigma,\rho}(z,w|\bar\alpha,\bar\beta)$, equation (2.48). Since the function $\mathcal{K}^m_{\lambda,\sigma,\rho}(z,w|\bar\alpha,\bar\beta)$ is an analytic function of $z,w,\bar\alpha,\bar\beta$ in the vicinity of the point $z=w=\alpha=\beta=1$ $(z_{ik}=\cdots=\beta_{ik}=0)$ it defines an operator on the tensor product of Verma modules $\mathbb{V}\otimes\mathbb{V}$. Thus, we have proven the following statement.

Lemma 2. *The operator* $\mathbb{R}^{(m)}_{12}(\lambda)$: $\mathbb{V}\times\mathbb{V}\to\mathbb{V}\times\mathbb{V}$ *defined by the kernel*

$$\mathcal{R}^{(m)}_{\lambda,\sigma\rho}(z,w|\bar\alpha,\bar\beta)=A_m\left(\bar\beta_w w^{-1}z\bar\alpha_z^{-1}\right)^\lambda_{mm}\mathcal{I}^\sigma(z,\alpha)\mathcal{I}^\rho(w,\beta), \tag{3.22}$$

where A_m *is some constant, solves the RLL relation (3.19).*

As follows from equation (3.22) the operator $\mathbb{R}^{(m)}$ depends both on the parameters $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$, and on the spectral parameter λ. We will display explicitly the dependence on a spectral parameter only. It means that a product of two operators, for instance $\mathbb{R}^{(m)}_{12}(\mu)\mathbb{R}^{(m)}_{12}(\lambda)$, written in an explicit form, turns into $\mathbb{R}^{(m)}_{12}(\mu|\boldsymbol{\sigma}',\boldsymbol{\rho}')\mathbb{R}^{(m)}_{12}(\lambda|\boldsymbol{\sigma},\boldsymbol{\rho})$, where the parameters $\boldsymbol{\sigma}',\boldsymbol{\rho}'$ are determined by equation (3.20).

It is easy to see from equation (3.22) that matrix elements of the operator $\mathbb{R}^{(m)}_{12}(\lambda)$ (modulo the prefactor A_m) are analytic functions of the spectral parameter λ. Moreover, they are meromorphic functions in $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$, which specify representations of the $sl(N)$ algebra on the space $\mathbb{V}\otimes\mathbb{V}$. The position of the poles corresponds to the points of reducibility of the representations $\pi^{\boldsymbol{\sigma}}$ and $\pi^{\boldsymbol{\rho}}$.

The normalization factor A_m in (3.22) is, in general, an arbitrary function of $\lambda,\boldsymbol{\sigma},\boldsymbol{\rho}$: $A_m = A_m(\lambda,\boldsymbol{\sigma},\boldsymbol{\rho})$. However, the operators $\mathbb{R}^{(m)}_{ik}$ satisfy the relations (2.38) only if the functions A_m obey some restrictions. They can easily be read off equations (2.38), for instance,

$$A_m(0,\boldsymbol{\sigma},\boldsymbol{\rho}) = 1, \qquad A_m(\mu,\boldsymbol{\sigma}',\boldsymbol{\rho}')A_m(\lambda,\boldsymbol{\sigma},\boldsymbol{\rho}) = A_m(\mu+\lambda,\boldsymbol{\sigma},\boldsymbol{\rho}) \tag{3.23}$$

and so on. The simplest normalization which satisfies all the requirements is $A_m = 1$. Choosing another, $SL(N,\mathbb{C})$ induced normalization, one can get rid of unessential prefactors in some formulae for transfer matrices. In this normalization the factor A_m reads

$$A_m(u-v,\boldsymbol{\sigma},\boldsymbol{\rho}) = f_m(u-v)\prod_{k=1}^{m-1}\frac{\Gamma(u_k-v_m+1)}{\Gamma(u_k-u_m+1)}\prod_{j=m+1}^{N}\frac{\Gamma(u_m-v_j+1)}{\Gamma(v_m-v_j+1)}, \tag{3.24}$$

where $f_m(\lambda) = 1$ for even m and $f_m(\lambda) = \mathrm{e}^{\mathrm{i}\pi\lambda}$ for odd m. We will assume that the normalization factor possesses all necessary properties. Its explicit form will be irrelevant for further discussion.

Let us prove that the operators $\mathbb{R}^{(m)}_{ik}(\lambda)$ satisfy the relations (2.38). The first of them, $\mathbb{R}^{(m)}_{ik}(0) = \mathbb{I}$, follows directly from equation (3.22). The second one, (2.38*b*), requires a special analysis and will be discussed in appendix B.

Going on to the proof of the relations (2.38*c*), (2.38*d*) and (2.38*e*) we note that the product of the operators $\mathbb{R}^{(m)}$ on the lhs and rhs of the corresponding equations results in the same permutations of the spectral parameters u_k, v_k, w_k in the product of two (three) Lax operators. Let us show that two operators which result in the same permutation of the spectral parameters in a product of Lax operators coincide up to a normalization. Namely, if

$$\begin{aligned} A(u-v)L_1(u)L_2(v) &= L_1'(u')L_2'(v')A(u-v),\\ B(u-v)L_1(u)L_2(v) &= L_1'(u')L_2'(v')B(u-v), \end{aligned} \tag{3.25}$$

then $A(\lambda) = cB(\lambda)$. It follows from equations (3.25) that the operator $C(\lambda) = B^{-1}(\lambda)A(\lambda)$ commutes with the product of Lax operators

$$C(u-v)L_1(u)L_2(v) = L_1(u)L_2(v)C(u-v). \tag{3.26}$$

We will assume that the operators $A(\lambda)$ and $B(\lambda)$ and, hence the operator $C(\lambda)$, are analytic operators which means that their matrix elements are analytic (meromorphic) functions of λ. The following lemma states that an operator with such properties is a multiple of the unit operator.

Lemma 3. *Let $\pi^{(1)}$ and $\pi^{(2)}$ be irreducible highest weight representations of the $sl(N)$ algebra on the spaces $\mathbb{V}_1$ and $\mathbb{V}_2$. If an analytic operator $A(\lambda):\mathbb{V}_1\otimes\mathbb{V}_2\to\mathbb{V}_1\otimes\mathbb{V}_2$ commutes with* *a product of Lax operators*

$$A(u-v)L_1(u)L_2(v) = L_1(u)L_2(v)A(u-v), \tag{3.27}$$

then $A(\lambda) = a(\lambda)\mathbb{I}$.

Proof. It follows from (3.27) that

$$\left[A(\lambda), E_{ki}^{(1)}(\lambda)\right] = \left[A(\lambda), E_{ki}^{(2)}(\lambda)\right] = 0, \tag{3.28}$$

where

$$E_{ki}^{(1)}(\lambda) = E_{ki}^{(1)} - \frac{1}{\lambda}\sum_m E_{km}^{(2)}E_{mi}^{(1)}, \qquad E_{ki}^{(2)}(\lambda) = E_{ki}^{(2)} + \frac{1}{\lambda}\sum_m E_{km}^{(2)}E_{mi}^{(1)}. \tag{3.29}$$

The space $\mathbb{V}\otimes\mathbb{V}$ can be decomposed into the direct sum of invariant subspaces of the Cartan generators, $\mathbb{V}\otimes\mathbb{V} = \sum_h \mathbb{V}_h$, $\dim \mathbb{V}_h = N_h < \infty$. Since $\left[E_{kk}^{(1)} + E_{kk}^{(2)}, A(u)\right] = 0$, the subspace $\mathbb{V}_h$ is an invariant subspace of the operator $A(u)$. Let e_n^h, $n = 1, \dots, N_h$, be a basis in the subspace $\mathbb{V}_h$. Since the representations $\pi^{(1)}, \pi^{(2)}$ are irreducible, the basis vectors e_n^h are given by linear combinations of the vectors

$$\prod_{k>i}\left(E_{ki}^{(1)}\right)^{n_{ki}}\prod_{j>m}\left(E_{jm}^{(2)}\right)^{n_{jm}}\upsilon_0,$$

where υ_0 is the highest weight vector, $\upsilon_0 = 1$. So we write $e_n^h = e_n^h(E^{(1)}, E^{(2)})$. Now let us consider a set of the vectors $e_n^h(\lambda) = e_n^h(E^{(1)}(\lambda), E^{(2)}(\lambda))$. It follows from equation (3.29) that for a sufficiently large λ, the vectors $e_n^h(\lambda) = e_n^h + \mathcal{O}(1/\lambda)$, $n = 1, \dots, N_h$, are linearly independent. Hence they form a basis in the subspace $\mathbb{V}_h$. By virtue of equation (3.28) one finds that $A(\lambda)e_n^h(\lambda) = a(\lambda)e_n^h(\lambda)$, where $a(\lambda) = A(\lambda)\upsilon_0$. Thus, we have proven that

$$A(\lambda) = a(\lambda)\mathbb{I} \tag{3.30}$$

on an arbitrary subspace $\mathbb{V}_h$ in some region of λ. Due to assumed analyticity equation (3.30) is valid for an arbitrary λ. Therefore equation (3.30) holds for the whole space $\mathbb{V}_1\otimes\mathbb{V}_2$. □

It is clear that the proof of lemma 3 can easily be extended to the case of an arbitrary number of Lax operators. Namely, if an analytic operator $A(\lambda_1, \lambda_2, \lambda_{M-1})$ satisfies the equation

$$A(\lambda_i)L_1(u)L_2(u+\lambda_1)\cdots L_M(u+\lambda_{M-1}) = L_1(u)L_2(u+\lambda_1)\cdots L_M(u+\lambda_{M-1})A(\lambda_i), \tag{3.31}$$

then $A(\lambda_i) \sim \mathbb{I}$. As was explained earlier this result implies that the lhs and rhs of equations (2.38*c*), (2.38*d*) and (2.38*e*) are equal to each other up to some factor, $\varkappa$. It can easily be checked by examining the action of the operators on the highest weight vector, $\upsilon_0 = 1$, that $\varkappa = 1$. We also note here that *lemma* implies the uniqueness of the solution of the RLL relation (3.19).

The commutation relations (2.38*b*), (2.40) for the operators $\mathbb{R}^{(m)}$ are vital for our analysis of transfer matrices, see section 4. The proof of these relations makes use of the explicit form of the factorizing operators, equation (3.22), and the invariance property of the bilinear form Ω_σ. First of all, we prove the following lemma.

Lemma 4. *Let* $g \in SL(N)$ *and* $\Delta_k(g) \neq 0$, $k = 1, \dots, N-1$. *Then*

$$\Omega_\sigma(\widetilde{T}^{-\sigma}(g^{-1})\bar{e}_m, e_n) = \Omega_\sigma(\bar{e}_m, T^\sigma(g)e_n). \tag{3.32}$$

Here $e_n(z)$ *and* $\bar{e}_m(w) = (e_m(w))^*$ *are basis vectors in the spaces* $\mathbb{V}, \overline{\mathbb{V}}$ *(see equation (3.5)). The transformations* $T^\sigma(g), \widetilde{T}^{-\sigma}(g^{-1})$ *are defined by equations (2.1), (2.19).*

Proof. Since $\Delta_k(g) \neq 0$ the functions $T^\sigma(g)e_n(z)$ and $\widetilde{T}^{-\sigma}(g^{-1})\bar{e}_m(z)$ are (anti)holomorphic functions in the vicinity of the point $z = 1$ $(z_{ik} = 0)$

$$[T^\alpha(g)e_n](z) = \sum_k A_{kn}e_k(z), \qquad [\widetilde{T}^\alpha(g^{-1})e_m](\bar{z}) = \sum_k \overline{e_k(z)}B_{mk}. \quad (3.33)$$

The matrices A and B satisfy the following relation:

$$(\Omega^{-1}B)_{nm} = (A\Omega^{-1})_{nm}, \quad (3.34)$$

which follows immediately from the identity for the reproducing kernel,

$$\left[T_z^\sigma(g)\mathcal{I}^\sigma\right](z, w) = \left[\widetilde{T}_w^{-\sigma}(g^{-1})\mathcal{I}^\sigma\right](z, w),$$

(see equation (2.22)). Inserting (3.33) into (3.32) one obtains that the latter reduces to equation (3.34). □

Further if both functions $Q(\bar{z})$ and $[\widetilde{T}^{-\sigma}(g^{-1})Q](\bar{z})$ are analytic at $z = 1$ then one obtains

$$\Omega_\sigma(\widetilde{T}^{-\sigma}(g^{-1})Q, e_m) = \sum_n q_n\Omega_\sigma(\bar{e}_n, T^\sigma(g)e_m), \quad (3.35)$$

where $Q(\bar{z}) = \sum_n q_n\overline{e_n(z)}$.

Lemma 5. *If the kernel of an operator A has the form*

$$A(z, \alpha) = r(z, \bar{\alpha}_z)\mathcal{I}^\sigma(z, \alpha), \quad (3.36)$$

where the function $r(z, \bar{\alpha})$ does not depend on the variables:

(a) α^*_{nj}, $j < m$ or
(b) $(\alpha^{-1})^*_{jn}$, $j > m$ then

(i) A commutes with z_{kj}, $j < m$, $Az_{kj} = z_{kj}A$,
(ii) A commutes with z^{-1}_{jk}, $j > m$, $Az^{-1}_{jk} = z^{-1}_{jk}A$.

Proof. The proofs for cases (a) and (b) are similar, so we consider case (a) only. Since the function $r(z, \bar{\alpha})$ depends only on a part of the variables α^*_{nj} its expansion in a power series

$$r(z, \bar{\alpha}) = \sum_{ij} r_{ij}e_i(z)\overline{e_j(\alpha)} \quad (3.37)$$

runs only over those basis vectors $\bar{e}_j$ which lie in the subspaces $\overline{\mathbb{V}}_h$ with the multi-index $h = (0, \ldots, 0, h_m, \ldots, h_{N-1})$. Therefore, taking into account equation (3.7) one derives

$$\Omega_\sigma(r(z, \bar{\alpha}), P(\alpha)) = \Omega_\sigma(r(z, \bar{\alpha}), \Pi_m P(\alpha)), \quad (3.38)$$

where Π_m is a projector to the subspace $\mathbb{V}_m = \sum_{h=(0,\ldots,0,h_m,\ldots,h_{N-1})} \oplus\mathbb{V}_h$,

$$[\Pi_m P](\alpha) = P(\alpha)|_{\alpha_{kj}=0, j<m}. \quad (3.39)$$

Noting that the kernel $A(z, \alpha)$ has the form $\widetilde{T}^{-\sigma}(z)r(z, \bar{\alpha})$ and making use of equation (3.35) one obtains

$$\begin{aligned} AP(z) &= \Omega_\sigma(A(z, \alpha), P(\alpha)) = \Omega_\sigma(\widetilde{T}^{-\sigma}(z)r(z, \bar{\alpha}), P(\alpha)) \\ &= \sum_{ij} r_{ij}e_i(z)\Omega_\sigma(\overline{e_j(\alpha)}, T^\sigma(z^{-1})P(\alpha)) = \sum_{ij} r_{ij}e_i(z)\Omega_\sigma(\overline{e_j(\alpha)}, \Pi_m T^\sigma(z^{-1})P(\alpha)). \end{aligned} \quad (3.40)$$

Taking $P(z) = z_{kj}\widetilde{P}(z)$, $j < m$ and noting that

$$\Pi_m T^\sigma(z^{-1})\alpha_{kj}\widetilde{P}(\alpha) = z_{kj}\Pi_m T^\sigma(z^{-1})\widetilde{P}(\alpha)$$

we obtain the necessary result. □

The commutation relations (2.39) for the operators $\mathbb{R}_{12}^{(m)}(\lambda)$ are a simple corollary of this lemma. The proof of the two remaining relations (2.40) and (2.38*b*) is given in appendix B.

Finally, with the help of the relations (2.39) one can check that the operator $\mathbb{R}^{(m)}(\lambda)$ depends only on a part of the parameters $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$ characterizing the Verma modules, namely

$$\mathbb{R}_{\sigma\rho}^{(m)}(\lambda) = \mathbb{R}^{(m)}(\lambda, \sigma_{12}, \ldots, \sigma_{m-1m}, \rho_{mm+1}, \ldots, \rho_{N-1N}). \tag{3.41}$$

Proving the properties (2.38), (2.39) and (2.40) we have assumed that the representations of the $sl(N)$ algebra the factorizing operators act on are irreducible. This condition can be relaxed. Indeed, all relations in question give rise (and are equivalent) to certain equations for matrix elements of the factorizing operators $\mathbb{R}^{(m)}$. The matrix elements depend analytically on the parameters specifying the representations. Thus these equations hold for arbitrary parameters and, hence, for reducible representations.

On the basis of the obtained results we conclude that the following theorem holds:

Theorem 1. *An $sl(N)$ invariant solution of the YBE on a tensor product of generic highest weight representations of the $sl(N)$ algebra can be represented in the factorized form*

$$\mathcal{R}_{12}(u-v) = P_{12}\mathbb{R}_{12}^{(1)}(u_1-v_1)\mathbb{R}_{12}^{(2)}(u_2-v_2)\cdots\mathbb{R}_{12}^{(N)}(u_N-v_N). \tag{3.42}$$

Here $u_i = u-\sigma_i$, $v_i = v-\rho_i$. The parameters $\boldsymbol{\sigma}$, $\boldsymbol{\rho}$ specify the representations $\pi^{(1)}$ and $\pi^{(2)}$, respectively. P_{12} is a permutation operator. The factorizing operators $\mathbb{R}_{12}^{(m)}(\lambda)$ are given by equation (3.22) and satisfy (at proper normalization) the relations (2.38), (2.39) and (2.40).

The Verma module is irreducible if none of the differences, $\sigma_i - \sigma_k$, $k > i$, is a positive integer. In other cases there exists an (in)finite-dimensional invariant subspace υ, $\upsilon \subset \mathbb{V}$. Let π' be a restriction of π^{σ} onto the subspace υ, $\pi' = \pi^{\sigma}|_{\upsilon}$, and π'' is a representation induced on the factor space $\mathbb{V}/\upsilon$. It follows from the RLL relation that the space $\upsilon\otimes\mathbb{V}$ (we assume that the representation π^{ρ} is irreducible) is an invariant subspace of the operator $\mathcal{R}_{12}$. Therefore the $\mathcal{R}$-matrix has a block-triangular form

$$\mathcal{R}_{12}(u) = \begin{pmatrix} \mathcal{R}'_{12}(u) & \star \\ 0 & \mathcal{R}''_{12}(u) \end{pmatrix}, \tag{3.43}$$

where the diagonal blocks, $\mathcal{R}'_{12}(u) = \mathcal{R}_{12}(u)|_{\upsilon\otimes\mathbb{V}}$ and $\mathcal{R}''_{12}(u) = \mathcal{R}_{12}(u)|_{(V/\upsilon)\otimes\mathbb{V}}$, define new $\mathcal{R}$ matrices on the spaces $\upsilon\otimes\mathbb{V}$ and $(V/\upsilon)\otimes\mathbb{V}$, respectively. Thus one can extract $\mathcal{R}$-matrices for arbitrary (non-generic) representations of the $sl(N)$ algebra by studying the $\mathcal{R}$-matrix (3.42) for reducible generic representations.

Let us put $u_m - v_m = \lambda$ and $u_k = v_k$, $k \neq m$ in equation (3.42). These constraints mean that the characters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in the tensor product $\pi^{\alpha}\otimes\pi^{\beta}$ are related. Namely, one easily finds that $\boldsymbol{\beta}(h) = \boldsymbol{\alpha}_{m,\lambda}(h) = h_{mm}^{-\lambda}\boldsymbol{\alpha}(h)$. Taking into account that $\mathbb{R}_{12}^{(k)}(0) = \mathbb{I}$ one derives

$$\mathcal{R}_{12}^{(m)}(\lambda) \equiv P_{12}\mathbb{R}_{12}^{(m)}(\lambda) = \mathcal{R}_{12}\left(\frac{\lambda}{N}\right)\Bigg|_{\beta=\alpha_{m,\lambda}}. \tag{3.44}$$

Constructing transfer matrices one refers to spaces in the tensor product $\mathbb{V}\otimes\mathbb{V}$ as a quantum (first) and auxiliary (second) spaces, respectively. Clearly, the operator $\mathcal{R}_{12}^{(m)}(\lambda)$ is completely fixed by the spectral parameter λ and the representation $\pi^{\alpha}(\equiv\pi^{\sigma})$ on the quantum space. Making use of equation (3.41) one gets

$$\mathcal{R}_{12}^{(m)}(\lambda) = P_{12}\mathbb{R}^{(m)}(\lambda, \sigma_{12}, \ldots, \sigma_{m-1m}, \lambda+\sigma_{mm+1}, \ldots, \lambda+\sigma_{N-1N}). \tag{3.45}$$

In the following section we study the properties of the operators defined as the trace of a monodromy matrices constructed from the operators $\mathcal{R}_{12}^{(m)}(\lambda)$.

4. Transfer matrices and Baxter $\mathcal{Q}$-operators

4.1. Transfer matrices

In this section we discuss properties of transfer matrices for generic $sl(N)$ spin chains. A transfer matrix is defined as a trace of a monodromy matrix which is given by the product of $\mathcal{R}$-operators. Since we consider infinite-dimensional representations of the $sl(N)$ algebra a convergence of the trace is not guaranteed. Except for the $N = 2$ case, in order to ensure the finiteness of traces one has to introduce a regulator (boundary operator) [15, 19, 24, 25] which breaks $sl(N)$ symmetry down to its diagonal subgroup. Namely, let us define a new $\mathcal{R}$-operator by

$$\mathcal{R}_{12}(u,\tau) \equiv \mathcal{R}_{12}(u,\tau_1,\ldots,\tau_{N-1}) = \prod_{p=1}^{N-1} \tau_p^{(H_2)_p}\mathcal{R}_{12}(u) \equiv \tau^{H_2}\mathcal{R}_{12}(u). \tag{4.1}$$

The operators H_p are defined in (3.3) and the index 2 (H_2) refers to the space the operator acts on. The operators $\mathcal{R}_{12}(u,\tau)$ obey the YBE

$$\mathcal{R}_{12}(u,\tau)\mathcal{R}_{13}(v,\tau)\mathcal{R}_{23}(v-u) = \mathcal{R}_{23}(v-u)\mathcal{R}_{13}(v,\tau)\mathcal{R}_{12}(u,\tau). \tag{4.2}$$

We define a transfer matrix by[8]

$$\mathsf{T}_\rho(u,\tau) = \mathrm{tr}_\rho\{\mathcal{R}_{10}(u,\tau)\ldots\mathcal{R}_{L0}(u,\tau)\}. \tag{4.3}$$

The index $\boldsymbol{\rho}$ specifies a representation of $sl(N)$ algebra, $\pi^{\rho}(=\pi^{\beta})$, on the auxiliary space. We will also label a transfer matrix by the highest weight of the representation in the auxiliary space $\boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_{N-1})$, $\lambda_k = 1-\rho_k+\rho_{k+1}$, i.e.

$$\mathsf{T}_{\boldsymbol{\lambda}}(u,\tau) \equiv \mathsf{T}_\beta(u,\tau) \equiv \mathsf{T}_\rho(u,\tau). \tag{4.4}$$

Let us note that for $\tau < 1$ the factor τ^{H_2} improves a convergence of the trace since the eigenvalues of the operators H_p, $p = 1,\ldots,N-1$ (see equation (3.3)) are positive integers. Thus, provided that the trace exists, equation (4.3) defines an operator on the tensor product of Verma modules $\mathbb{V}_1\otimes\cdots\otimes\mathbb{V}_L$. The transfer matrices form a commutative family of operators

$$\left[\mathsf{T}_{\rho_1}(u,\tau),\mathsf{T}_{\rho_2}(v,\tau)\right] = 0. \tag{4.5}$$

It is easy to see that the transfer matrix $\mathsf{T}_\rho(u,\tau)$ commutes also with the total Cartan generators, namely $[\mathsf{T}_\rho(u,\tau),H_p] = 0$, $p = 1,\ldots,N-1$, where $H = H^{(1)}+\cdots+H^{(L)}$.

We will consider homogeneous spin chains only, i.e. assume that the representations of $sl(N)$ algebra on the quantum space at each site are equivalent, $\pi^{\sigma_1} = \pi^{\sigma_2} = \cdots = \pi^{\sigma_L} \equiv \pi^{\sigma}$.

4.2. Baxter Q-operators

Let us define special transfer matrices, the Baxter Q-operators, constructed from the operators $\mathcal{R}^{(m)}$, (3.44). Namely, similarly to (4.1) we put

$$\mathcal{R}_{12}^{(m)}(u,\tau) = \tau^{H_2}\mathcal{R}_{12}^{(m)}(u) \tag{4.6}$$

and

$$\mathcal{Q}_k(u+\sigma_k,\tau) = \mathrm{tr}_0\left\{\mathcal{R}_{10}^{(k)}(u,\tau)\ldots\mathcal{R}_{L0}^{(k)}(u,\tau)\right\}. \tag{4.7}$$

[8] In [24] the transfer matrix with an insertion of a boundary operator was defined as follows: $\widetilde{T}(u,\tau) = \mathrm{tr}_\rho(\tau^{H_0}R_{10}(u)\cdots R_{L0}(u))$. It is easy to check that these two definitions are essentially the same $T(u,\tau) = \tau^A\widetilde{T}(u,\tau^L)\tau^{-A}$, where $A = \sum_{k=1}^{L-1}(L-k)H_k$.

Taking into account equation (3.44) one finds that $\mathcal{Q}_k(u)$ is given by the transfer matrix for a special choice of the auxiliary space

$$\mathcal{Q}_k(u+\sigma_k,\tau) = \mathsf{T}_{\alpha_{k,u}}(u/N,\tau) = \mathsf{T}_{\boldsymbol{\lambda}-u(e_k-e_{k-1})}(u/N,\tau), \tag{4.8}$$

where the vector e_k is defined by $(e_k)_n = \delta_{kn}$, and $\boldsymbol{\lambda}$ is the highest weight of the representation in the quantum space, $\lambda_k = \sigma_{k+1} - \sigma_k + 1$.

The Baxter operators $\mathcal{Q}_k(u,\tau)$ act on the quantum space of a model and commute with each other

$$[\mathcal{Q}_k(u,\tau), \mathcal{Q}_m(v,\tau)] = 0. \tag{4.9}$$

One easily finds that the Baxter operators satisfy the following normalization condition:

$$\mathcal{Q}_k(\sigma_k,\tau) = \mathcal{P}\tau^H, \tag{4.10}$$

where $H = H^{(1)} + \cdots + H^{(L)}$ and $\mathcal{P}$ is the operator of cyclic permutation,

$$\mathcal{P} f(z_1,\ldots,z_L) = f(z_L, z_1,\ldots,z_{L-1}). \tag{4.11}$$

Let us note also that $[\mathcal{Q}_k(u),\mathcal{P}] = [\mathcal{Q}_k(u), H] = 0$.

Now we are going to prove that equation (4.7) gives rise to a well-defined operator on the quantum space. To this end it is necessary to show that the trace over an infinite-dimensional auxiliary space converges. Let $\mathbb{Q}_k(u,\tau)$ be a monodromy matrix,

$$\mathbb{Q}_k(u,\tau) = \mathcal{R}^{(k)}_{10}(u,\tau)\ldots\mathcal{R}^{(k)}_{L0}(u,\tau), \tag{4.12}$$

and $[\mathbb{Q}_k(u,\tau)]^{m'_1,\ldots,m'_L,n'}_{m_1,\ldots,m_L,n}$ its matrix elements in the basis $E_{\vec{m},n}$,

$$E_{\vec{m},n} = e_n(w)\prod_{k=1}^{L} e_{m_k}(z^{(k)}),$$

$$\mathcal{Q}_k(u\tau)E_{\vec{m},n} = [\mathbb{Q}_k(u,\tau)]^{m'_1,\ldots,m'_L,n'}_{m_1,\ldots,m_L,n} E_{\vec{m}',n'}.$$

Here we keep the variable w for the auxiliary space and $z^{(k)}$ for the quantum space in the kth site. The basis vectors $e_n(z)$ are defined by equation (3.5).

The trace of the monodromy matrix $\mathbb{Q}_k(u,\tau)$ written explicitely takes the form

$$\sum_{\boldsymbol{n}=(n_1,n_2,\ldots,n_L)} \left[\mathcal{R}^{(k)}_{10}\right]^{m'_1 n_L}_{m_1,n_1}\left[\mathcal{R}^{(k)}_{20}\right]^{m'_2 n_1}_{m_2,n_2}\cdots\left[\mathcal{R}^{(k)}_{L0}\right]^{m'_L n_{L-1}}_{m_L,n_L}. \tag{4.13}$$

We recall that each summation index, n_k is multi-index, $n_k = \{(n_k)_{ij}, i > j\}$. Let us introduce notations, $\boldsymbol{n} = (n_1, n_2,\ldots,n_L)$, and $\boldsymbol{n}_{ij} = ((n_1)_{ij}, (n_2)_{ij},\ldots,(n_L)_{ij})$. First of all, we show that all summation indices $\boldsymbol{n}_{ij}$ except $\boldsymbol{n}_{k+1k},\ldots,\boldsymbol{n}_{Nk}$ vary in a finite range, while the indices $\boldsymbol{n}_{k+1k},\ldots,\boldsymbol{n}_{Nk}$ can be arbitrarily large. To this end we examine an action of the operator $\mathcal{R}^{(k)}_{l0}$ on the basis vector $e_{m_l}(z_l)\otimes e_{n_l}(w)$. For brevity we skip the index l, i.e. $z_l\to z$, $m_l\to m$, etc. Then taking into account equations (2.39) one finds

$$\begin{aligned}\mathcal{R}^{(k)}_{l0} e_m(z)\otimes e_n(w) &= P_{zw}\left(\prod_{j<k,i} w_{ij}^{n_{ij}}\right)\mathbb{R}^{(k)}_{l0}\left(e_m(z)\otimes\prod_{i>j\geqslant k} w_{ij}^{n_{ij}}\right)\\ &= \left(\prod_{j<k,i} z_{ij}^{n_{ij}}\right)\mathcal{R}^{(k)}_{l0}\left(e_m(z)\otimes\prod_{i>j\geqslant k} w_{ij}^{n_{ij}}\right).\end{aligned} \tag{4.14}$$

From here one concludes that the indices n_{ij} for $j < k$ are restricted from above by m'_{ij}, $n_{ij}\leqslant m'_{ij}$, i.e. $\boldsymbol{n}_{ij}\leqslant \boldsymbol{m}'_{ij}$ for $j<k$.

To prove that $\boldsymbol{n}_{ij}$ is restricted for $j > k$ we use the relation (2.40). It takes the form

$$D^{w}_{p+1,p}\mathcal{R}^{(k)}_{l0} = \mathcal{R}^{(k)}_{l0} D^{z}_{p+1,p}, \qquad p > k. \tag{4.15}$$

Since for a given m the operators $\left(D^{z}_{p+1,p}\right)^{M_p}$, where M_p is some number, nullify the vector $e_m(z)$ one derives that $\left(D^{w}_{p+1,p}\right)^{M_p} E_{mn}(z,w) = 0$ for $p > k$, where $E_{mn}(z,w) = \mathcal{R}^{(k)}_{l0}(e_m(z)\otimes e_n(w))$. The function E_{mn} satisfying these conditions is a polynomial of finite degree (which depends on M_p) in w_{ip}, $p > k$ (see [44, chapter X]) hence, $\boldsymbol{n}_{ij} \leqslant \boldsymbol{M}'_{ij}$, for $j > k$.

Thus we have shown that the summation over $n_p = \{(n_p)_{ij}\}$, $p = 1,\dots,L$ in (4.13) goes in a finite range for all $(n_p)_{ij}$ except $(n_p)_{k+1,k},\dots,(n_p)_{N,k}$. Let us also note that all summation indices, $(n_p)_{ik}$ are of the same order, the difference $(n_p)_{ik} - (n_L)_{ik} = (q_p)_{ik}$ being finite when $(n_L)_{ik}$ goes to infinity. It will be shown in appendix C that for the matrix element $[\mathcal{R}^{(k)}_{l0}(u)]^{m'n'}_{mn}$ in the limit $n_{ik} \to \infty$, $i = k+1,\dots,N$, all other variables being fixed, the following estimate holds:

$$\left|\left[\mathcal{R}^{(k)}_{l0}(u)\right]^{m'n'}_{mn}\right| < C(u) h^{a_{k+1}}_{k+1}\cdots h^{a_N}_{N}, \tag{4.16}$$

where $h_k = \sum_{i=k+1}^{N} n_{ik} + 1$ and $C(u), a_k,\dots,a_N$ are some constants, which depends on $\boldsymbol{\sigma}, m_{ij},\dots$. Since $\tau^{H_2} \sim \tau_k^{h_k}\cdots\tau_{N-1}^{h_{N-1}}$ the estimate (4.16) ensures that the series in (4.13) converges absolutely $\boldsymbol{\tau} < 1$. Thus we have proven that equation (4.7) provides a definition of an operator on the tensor product of Verma modules.

Let us note that the trace for the operator $\mathcal{Q}_N$ is given by a finite sum. Therefore, the Baxter operator $\mathcal{Q}_N(u,\tau)$ has a finite limit at $\tau \to 1$. The operators $\mathcal{Q}_k(u,\tau)$, $k < N$, could be singular in this limit. However, as it follows from the above discussion, the operator $\mathcal{Q}_k(u,\tau)$ has a finite limit at $\tau_i \to 1$, $i \neq k$ and $\tau_k < 1$ is fixed.

4.3. Factorized form of transfer matrix

The transfer matrix (4.3) for a chosen quantum space depends on N complex parameters: the spectral parameter u and $N-1$ parameters, $\rho_k - \rho_{k+1}$, $k = 1,\dots,N-1$, specifying the representation on auxiliary space. The Baxter $\mathcal{Q}$-operators depend only on a spectral parameter. The number of independent parameters in the transfer matrix matches the number of spectral parameters in a product of N Baxter operators. Below we show that the following statement holds:

Theorem 2. *The transfer matrix $\mathsf{T}_\rho(u,\tau)$, $\tau < 1$, is factorized into the product of the Baxter $\mathcal{Q}$-operators*

$$\mathsf{T}_\rho(u,\tau) = \mathcal{Q}_1(u+\rho_1,\tau)(\mathcal{P}\tau^{H})^{-1}\mathcal{Q}_2(u+\rho_2,\tau)\cdots(\mathcal{P}\tau^{H})^{-1}\mathcal{Q}_N(u+\rho_N,\tau), \tag{4.17}$$

where $H = H^{(1)} + \cdots + H^{(L)}$.

Proof. The proof of (4.17) relies on the commutation relations (2.38*c*). Let $\mathsf{T}_k(u,\tau)$ be the transfer matrix for a special choice of the auxiliary space

$$\beta_k = \prod_{j=1}^{k} \Delta_j^{1-\rho_{jj+1}} \Delta_k^{1-\rho_k+\sigma_k-u} \prod_{j=k+1}^{N-1} \Delta_j^{1-\sigma_{jj+1}},$$

where $\boldsymbol{\sigma} = (\sigma_1,\dots,\sigma_N)$ are the parameters specifying the representation on the quantum space.

The transfer matrix $\mathsf{T}_k(u,\tau)$ can be represented in the form

$$\mathsf{T}_k(u,\tau) = \mathrm{tr}_{\beta_k}\mathcal{R}^{(1k)}_{10}(u,\tau)\cdots\mathcal{R}^{(1k)}_{L0}(u,\tau). \tag{4.18}$$

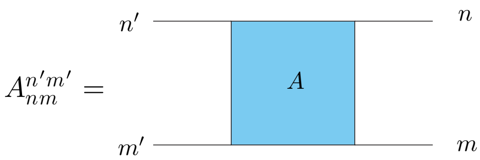


Figure 1. The graphical representation for the matrix element of an operator.

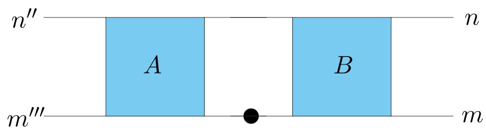


Figure 2. The graphical representation for the sum $\sum_{n',m',m''} A^{n''m'''}_{n'm''}[\tau^H]^{m''}_{m'} B^{n'm'}_{nm}$.

The reduced operator $\mathcal{R}^{(1k)}$ is defined as follows:

$$\mathcal{R}^{(1k)}_{j0}(u-v,\tau) = \tau^{H_0} P_{j0}\mathbb{R}^{(1)}_{j0}(u_1-v_1)\cdots\mathbb{R}^{(k)}_{j0}(u_k-v_k) \equiv \tau^{H_0} P_{j0}\mathbb{R}^{(1k)}_{j0}(u-v,\tau), \tag{4.19}$$

where $u_j = u - \sigma_j$, $v_j = v - \rho_j$. Let us show that

$$\mathsf{T}_k(u,\tau) = \mathsf{T}_{k-1}(u,\tau)(\mathcal{P}\tau^H)^{-1}\mathcal{Q}_k(u+\rho_k,\tau). \tag{4.20}$$

Obviously, the factorization formula (4.17) is a simple corollary of this result. To prove equation (4.20) let us put $A_j = \mathbb{R}^{(1k-1)}_{j0}(u-v)$ and $B_j = \mathbb{R}^{(k)}_{j0}(u_k-v_k)$, i.e.

$$\mathcal{R}^{(1k)}_{j0}(u-v,\tau) = \tau^{H_0} P_{j0} A_j B_j. \tag{4.21}$$

Matrix elements of the operators on both sides of equation (4.20) are given by some sums and one has to show that they are equal. To this end it is convenient to use a graphical representation for the sums. Let us denote the matrix element $A^{n'm'}_{nm}$ of the operator $A_j(B_j)$ by a box with four legs as shown in figure 1. The line connecting two boxes will imply a summation over the corresponding index. The operators τ^H (τ^{-H}) will be denoted by an insertion of black (white) circle in the corresponding line. As an example, we have given the diagrammatic representation for the sum $\sum_{n',m',m''} A^{n''m'''}_{n'm''}[\tau^H]^{m''}_{m'} B^{n'm'}_{nm}$ in figure 2.

The graphical representations for the rhs and lhs of equation (4.20) are shown in figures 3 and 4, respectively. To obtain the diagram shown in figure 3 we have used the commutativity of the Baxter operators with the total diagonal generators, $H = H^{(1)} + \cdots + H^{(L)}$ and represented the rhs of equation (4.20) as $T_{k-1}(u,\tau)\mathcal{P}^{-1}\mathcal{Q}_k(u+\rho_k,\tau)\tau^{-H}$.

We have already shown that the trace in equation (4.7) converges absolutely for $\tau < 1$. Let us assume now that the factorization formula (4.20) holds for the transfer matrices $\mathsf{T}_k(u,\rho)$, $k = 2,\ldots,p-1$ and that the traces for $\mathsf{T}_k(u,\rho)$, $k \leqslant p-1$ converge absolutely. Then it can be shown that equation (4.20) holds for $k = p$ and the corresponding trace converges absolutely. First of all, let us note that the summation over indices $k_1,\ldots,k_L$ in

$$\sum_{k_1,\ldots,k_L} [\mathsf{T}_{k-1}]^{m'_1,\ldots,m'_L}_{k_1,\ldots,k_L} [\mathcal{P}^{-1}Q_k\tau^{-H}]^{k_1,\ldots,k_L}_{m_1,\ldots,m_L}$$

goes in a finite range. The blocks A_j and B_{j+1} can be interchanged with the help of the commutation relation (2.38c) whose graphical form is shown in figure 5. Using this identity

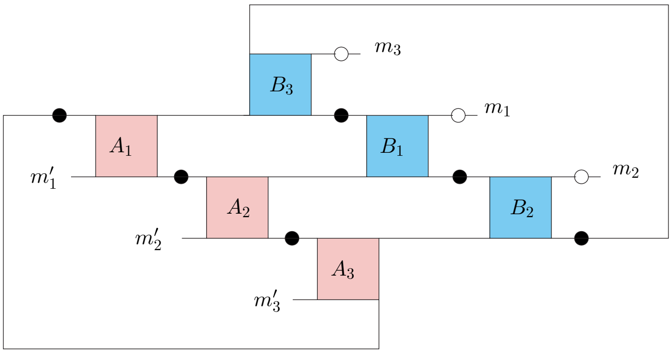


Figure 3. The graphical representation for the reduced transfer matrix T_k, the rhs of equation (4.18).

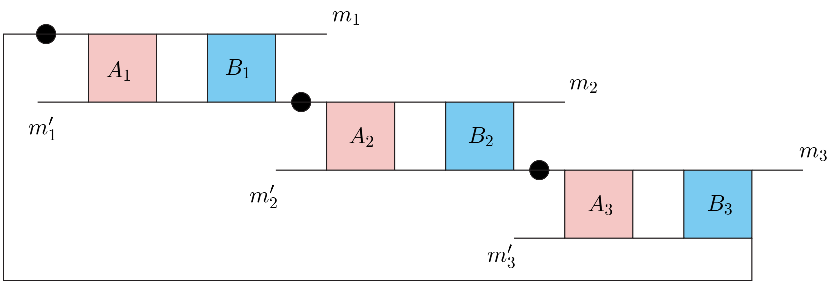


Figure 4. The graphical representation for the reduced transfer matrix T_k, the lhs of equation (4.18).

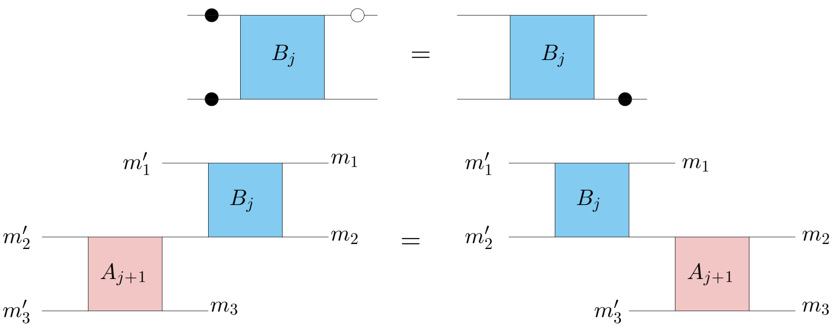
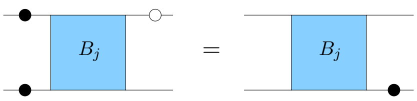


Figure 5. The graphical representation of the permutation identity (2.38c).

and taking into account that $\tau^{H_j+H_0}B_j\tau^{-H_j} = B_j\tau^{H_0}$



one can transform the sum depicted by the diagram in figure 3 into the sum in figure 4.

It is clear that the trace for T_k will converge absolutely, if it is the case for T_{k-1} and $\mathcal{Q}_k$. Since $\mathsf{T}_1(u, \tau) = \mathcal{Q}_1(u + \rho_1, \tau)$ the trace for T_1 converges absolutely. Therefore the factorization formula (4.20) and the absolute convergence of the traces for T_k for $k > 1$ will follow by induction over k. This completes the proof of the theorem. □

4.4. Fusion relations

It was long known that the transfer matrices satisfy a set of functional relations which are usually referred to as fusion relations. The fusion relations for the compact spin chains were thoroughly studied in the literature, see e.g. [45–51]. Below we show how the representation (4.17) can be used to obtain some functional relations for the transfer matrices.

Let us consider the product of two transfer matrices, $\mathsf{T}_\rho(u,\tau)$ and $\mathsf{T}_\omega(v,\tau)$. Both of them can be represented in the factorized form (4.17). Since all operators in (4.17) commute with each other one can interchange the Baxter operators $\mathcal{Q}_k(u+\rho_k)$ and $\mathcal{Q}_k(v+\omega_k)$. The new products can be identified as the transfer matrices, e.g.

$$(\mathcal{P}\tau^{\mathcal{H}})^{-(N-1)}\mathcal{Q}_k(v+\omega_k)\prod_{j\neq k}\mathcal{Q}_j(u+\rho_j)=\mathsf{T}_{\rho'}(u',\tau), \tag{4.22}$$

where $u'=u-\delta/N$, $\rho'_j=\rho_j+\delta/N$, for $j\neq k$ and $\rho'_k=\rho_k-\delta\,(1-1/N)$, $\delta=u-v+\rho_k-\omega_k$. Therefore one obtains the following relations:

$$\mathsf{T}_\rho(u,\tau)\mathsf{T}_\omega(v,\tau)=\mathsf{T}_{\rho'}(u-\delta/N,\tau)\mathsf{T}_{\omega'}(v+\delta/N,\tau), \tag{4.23}$$

where $\omega'_j=\omega_j-\delta/N, j\neq q$ and $\omega'_k=\omega_k+\delta(1-1/N)$. Changing the notation $\mathsf{T}_\rho(u,\tau)\to\mathsf{T}_{\boldsymbol{\lambda}}(u,\tau)=\mathsf{T}_{\lambda_1,\ldots,\lambda_{N-1}}(u,\tau)$ $(\mathsf{T}_\omega(v,\tau)\to\mathsf{T}_{\boldsymbol{\mu}}(v,\tau))$ where $\boldsymbol{\lambda}$ $(\boldsymbol{\mu})$ is the highest weight in the auxiliary space, $\lambda_k=\rho_{k+1}-\rho_k+1$, one rewrites relation (4.23) in the form

$$\mathsf{T}_{\boldsymbol{\lambda}}(u,\tau)\mathsf{T}_{\boldsymbol{\mu}}(v,\tau)=\mathsf{T}_{\boldsymbol{\lambda}-\delta_k(\boldsymbol{e}_{k-1}-\boldsymbol{e}_k)}(u-\delta_k/N,\tau)\mathsf{T}_{\boldsymbol{\mu}+\delta_k(\boldsymbol{e}_{k-1}-\boldsymbol{e}_k)}(v+\delta_k/N,\tau), \tag{4.24}$$

where $\boldsymbol{e}_k$ is $(N-1)$-dimensional vector $(\boldsymbol{e}_k)_i=\delta_{ik}$ and

$$\delta_k=u-v+\sum_{p=k}^{N-1}(\mu_p-\lambda_p)+\frac{1}{N}\sum_{p=1}^{N-1}p(\mu_p-\lambda_p). \tag{4.25}$$

The fusion relations (4.23), (4.24) remain valid for inhomogeneous spin chains.

4.5. Inhomogeneous spin chains

Let us explore modifications which appear in a general case of inhomogeneous spin chains with impurities (for the $sl(2)$ case see [35]). The transfer matrix is defined as

$$\mathsf{T}_\rho(u,\tau)=\mathrm{tr}_\rho\,\mathcal{R}_{10}(u+\xi_1,\tau)\mathcal{R}_{20}(u+\xi_2,\tau)\cdots\mathcal{R}_{L0}(u+\xi_L,\tau), \tag{4.26}$$

where $\{\xi_1,\ldots,\xi_L\}$ are impurity parameters and the quantum space in the kth site carries the representation $\pi^{(k)}$ of the $sl(N)$ algebra.

As in the case of the homogeneous spin chain the transfer matrix (4.26) can be represented in the form (4.17). However, properties of the factorizing $\mathcal{Q}$-operators change drastically. First of all, let us note that in the case of an inhomogeneous chain it is not possible to choose an auxiliary space representation, π^ρ, such that the operators $\mathcal{R}^k_{n0}(u)$ map $\pi^\rho\otimes\pi^{\sigma_n}$ onto itself, simultaneously for all n. Therefore, the quantum numbers of the auxiliary spaces at different sites are different.

Let us consider the monodromy matrix constructed from the operators $\mathcal{R}^{(k)}_{\sigma_j\rho_j}(u,\tau)$

$$\mathbb{Q}_k(u,\tau)=\mathcal{R}^{(k)}_{\sigma_1\rho_1}(u+\zeta_1,\tau)\cdots\mathcal{R}^{(k)}_{\sigma_L\rho_L}(u+\zeta_L,\tau), \tag{4.27}$$

where $\boldsymbol{\sigma}_j$ and $\boldsymbol{\rho}_j$ are the quantum numbers of the quantum and auxiliary spaces at the jth site and ζ_j are the impurities. In general, the monodromy matrix intertwines the representations with different quantum numbers

$$\mathbb{Q}_k(u,\tau)\pi^\rho\otimes\pi^{\sigma_1}\otimes\cdots\otimes\pi^{\sigma_L}=\pi^{\tilde\rho}\otimes\pi^{\tilde\sigma_1}\otimes\cdots\otimes\pi^{\tilde\sigma_L}\mathbb{Q}_k(u,\tau), \tag{4.28}$$

where we put $\boldsymbol{\rho} \equiv \boldsymbol{\rho}_L$. To be precise, for $\tau \neq 1$ the above relation holds only for the generators from the Cartan subalgebra. Provided that the representations π^{ρ} and $\pi^{\tilde{\rho}}$ coincide ($\boldsymbol{\rho} = \tilde{\boldsymbol{\rho}}$), the trace of the monodromy matrix $\mathsf{Q}_k(u,\tau) = \mathrm{tr}\,\mathbb{Q}_k(u,\tau)$ intertwines the Cartan generators of the representations $\pi^{\sigma_1} \otimes \cdots \otimes \pi^{\sigma_L}$ and $\pi^{\tilde{\sigma}_1} \otimes \cdots \otimes \pi^{\tilde{\sigma}_L}$. The condition $\boldsymbol{\rho} = \tilde{\boldsymbol{\rho}}$ fixes the quantum numbers, $\boldsymbol{\rho}_n$, of all auxiliary spaces in (4.27). We recall that

$$\mathbb{R}^k_{10}(u) : \pi^{\alpha} \otimes \pi^{\beta} \to \pi^{\alpha_{k,u}} \otimes \pi^{\beta_{k,-u}},$$

with $\boldsymbol{\alpha}_{k,u}(h) = h^{-u}_{kk}\boldsymbol{\alpha}(h)$, $\boldsymbol{\beta}_{k,-u}(h) = h^{u}_{kk}\boldsymbol{\beta}(h)$ (correspondingly, $\mathcal{R}^{(k)}_{10}(u) = P_{12}\mathbb{R}^k_{10}(u)$ maps $\pi^{\alpha} \otimes \pi^{\beta} \to \pi^{\beta_{k,-u}} \otimes \pi^{\alpha_{k,u}}$). Thus the parameter $\boldsymbol{\rho}_n$ in equation (4.27) is determined by the characters $(\boldsymbol{\alpha}_{n-1})_{k,u+\zeta_{n-1}}$. In its turn the quantum numbers $\tilde{\boldsymbol{\sigma}}_n$, equation (4.28), correspond to the character

$$\tilde{\boldsymbol{\alpha}}_n(h) = h^{\zeta_n - \zeta_{n+1}}_{kk}\boldsymbol{\alpha}_{n+1}(h).$$

Let us note that the character $\tilde{\boldsymbol{\alpha}}_n(h)$ does not depend on the spectral parameter u and is determined by the impurity parameters ζ only. Since one can always assume that $\sum \zeta_k = 0$ the operator $\mathsf{Q}_k(u,\tau)$ is uniquely determined by two sets of parameters, $\Sigma = (\boldsymbol{\sigma}_1, \ldots, \boldsymbol{\sigma}_L)$ and $\widetilde{\Sigma} = (\tilde{\boldsymbol{\sigma}}_1, \ldots, \tilde{\boldsymbol{\sigma}}_L)$,

$$\mathsf{Q}_k(u,\tau|\zeta,\Sigma) = \mathsf{Q}_k(u,\tau|\widetilde{\Sigma},\Sigma). \tag{4.29}$$

We will not display spins Σ, $\widetilde{\Sigma}$ assuming always that the operators are multiplied in a covariant way

$$\mathsf{Q}_k(u,\tau)\mathsf{Q}_j(v,\tau) = \mathsf{Q}_k(u,\tau|\widetilde{\widetilde{\Sigma}},\widetilde{\Sigma})\mathsf{Q}_j(v,\tau|\widetilde{\Sigma},\Sigma). \tag{4.30}$$

Having put the inpurity parameters in equation (4.27) to $\zeta_j = \xi_j - (\sigma_j)_k$ we, finally, define the operator Q_k as follows:

$$\mathsf{Q}_k(u,\tau) = \mathrm{tr}\,\mathcal{R}^{(k)}_{\sigma_1\rho_1}(u+\xi_1-(\sigma_1)_k,\tau)\cdots\mathcal{R}^{(k)}_{\sigma_L\rho_L}(u+\xi_L-(\sigma_L)_k,\tau), \tag{4.31}$$

where the representation $\pi^{\rho_n} \equiv \pi^{\beta_n}$ on the auxiliary space at the nth site corresponds to the character $\boldsymbol{\beta}_n = (\boldsymbol{\alpha}_{n-1})_{k,u+\xi_{n-1}-(\sigma_{n-1})_k}$. Quite similar to a homogeneous spin chain one can show that the transfer matrix for an inhomogeneous spin chain can be represented in the factorized form

$$\mathsf{T}_{\rho}(u,\tau) = \tau^{-(N-1)H}\mathsf{Q}_1(u+\rho_1,\tau)\mathcal{P}^{-1}\mathsf{Q}_2(u+\rho_2,\tau)\mathcal{P}^{-1}\cdots\mathcal{P}^{-1}\mathsf{Q}_N(u+\rho_N,\tau). \tag{4.32}$$

The operators $\mathsf{Q}_k(u,\tau)$ satisfy the following relations[9],

$$\mathsf{Q}_k(u,\tau)\mathsf{Q}_k(v,\tau) = \mathsf{Q}_k(v,\tau)\mathsf{Q}_k(u,\tau), \tag{4.33}$$

$$\mathsf{Q}_k(u,\tau)\mathcal{P}^{-1}\mathsf{Q}_n(v,\tau) = \mathsf{Q}_n(v,\tau)\mathcal{P}^{-1}\mathsf{Q}_k(u,\tau), \qquad \text{for} \qquad k \neq n, \tag{4.34}$$

which follow from the properties of the factorizing operators, equations (2.38c). These relations allow us to rearrange the Q_k operators in the product of T matrices in an arbitrary order. However, the operator Q_k alone cannot be considered as a 'good' operator on the quantum space of the model. Only the product of all operators Q_k has the necessary invariance properties. Therefore, it is reasonable to identify the Baxter $\mathcal{Q}_k$-operator with the transfer matrix (4.32), where only one operator Q_k depends on a spectral parameter. Namely, let $\boldsymbol{w} = (w_1, \ldots, w_N)$ and $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_{N-1})$, where $\mu_k = w_{k+1} - w_k + 1$. We define the Baxter operator as follows:

$$\begin{aligned}\mathcal{Q}^{(w)}_k(u+w_k,\tau) &= \mathsf{T}_{\mu-u(e_k-e_{k-1})}\left(\frac{u}{N},\tau\right)\\ &= \tau^{-(N-1)H}\mathsf{Q}_1(w_1,\tau)\mathcal{P}^{-1}\cdots\mathsf{Q}_k(w_k+u,\tau)\mathcal{P}^{-1}\cdots\mathcal{P}^{-1}\mathsf{Q}_N(w_N,\tau).\end{aligned} \tag{4.35}$$

[9] It should be noted that, despite appearance, equation (4.33) is not a commutation relation. Indeed, if written explicitly equation (4.33) becomes $\mathsf{Q}_k(u,\tau|\widetilde{\widetilde{\Sigma}},\widetilde{\Sigma})\mathsf{Q}_k(v,\tau|\widetilde{\Sigma},\Sigma) = \mathsf{Q}_k(v,\tau|\widetilde{\widetilde{\Sigma}},\Sigma')\mathsf{Q}_k(u,\tau|\Sigma',\Sigma)$.

The definition is not unique in a sense that the Baxter operator depends on the arbitrary parameters $\boldsymbol{w}$ ($\boldsymbol{\mu}$). In the case of the homogeneous spin chain there is a distinguished choice, $\boldsymbol{w} = \boldsymbol{\sigma}$, which ensures that $\mathcal{Q}_k(\sigma_k) = \mathcal{P}\tau^H$. However, it is clear that other variants are also feasible[10].

New operators, $\mathcal{Q}_k^{(\boldsymbol{w})}(u,\tau)$, acting on the quantum space of the model, form a commutative operator family. The normalization condition (4.10) which holds in a homogeneous case is now replaced by

$$\mathcal{Q}_k^{(\boldsymbol{w})}(w_k,\tau) = \mathsf{T}_w(0,\tau) \equiv \mathcal{Z}_w. \tag{4.36}$$

Using the commutation relations (4.33) it is straightforward to derive that the operators $\mathcal{Q}_k^{(\boldsymbol{w})}(u,\tau)$ possess all the properties which hold in a homogeneous case. Namely, the generic transfer matrix factorizes into the product of Baxter operators

$$\mathsf{T}_\rho(u,\tau) = \mathcal{Z}_w^{1-N}\mathcal{Q}_1^{(\boldsymbol{w})}(u+\rho_1,\tau)\cdots\mathcal{Q}_N^{(\boldsymbol{w})}(u+\rho_N,\tau). \tag{4.37}$$

The fusion relation for the generic transfer matrices has exactly the same form as in the homogeneous case, see equation (4.24).

5. Summary

In this paper we have developed an approach which allows us to construct the Baxter $\mathcal{Q}$-operators for a generic $sl(N)$ spin chain. We have proven that the $sl(N)$ invariant $\mathcal{R}$-operator on a tensor product of Verma modules can be represented in a factorized form. The factorizing operators have an extremely simple form (3.22) and possess a number of remarkable properties (2.38*c*). For the homogeneous spin chains we have defined the Baxter $\mathcal{Q}$-operators as the trace of the monodromy matrix constructed of the factorizing operators. We have shown that the Baxter $\mathcal{Q}$-operators can be identified with the transfer matrices for the special choice of the auxiliary space. This definition of the Baxter $\mathcal{Q}$-operators (see equation (4.35)) holds for inhomogeneous spin chains with impurities as well.

Many of the properties of the Baxter $\mathcal{Q}$-operators and transfer matrices follow readily from the properties of the factorizing operators, equations (2.38*c*), (2.39). In particular, we have shown that the generic transfer matrix is factorized into the product of *N* different Baxter $\mathcal{Q}$-operators. This representation for the transfer matrix together with the commutativity of the Baxter $\mathcal{Q}$-operators results immediately in certain functional relations for transfer matrices.

Another type of fusion relations involves the transfer matrices with a finite-dimensional auxiliary space. We recall that the Verma module $\mathbb{V}_\rho$ has an invariant finite-dimensional submodule υ_ρ if $\lambda_k = \rho_{k+1} - \rho_k + 1 = -n_k \leqslant 0, k = 1,\ldots,N-1$. (This representation corresponds to the Young tableau specified by the partition $\{\ell_1,\ell_2,\ldots,\ell_{N-1}\}$, where $\ell_k = \sum_{i=k}^{N-1} n_i$ is the length of the *k*th row in the tableau.) Let $t_\rho(u,\tau)$ be a trace of the monodromy matrix over such a finite-dimensional space

$$t_\rho(u,\tau) = \mathrm{tr}_{\upsilon_\rho}\mathcal{R}_{10}(u,\tau)\cdots\mathcal{R}_{L0}(u,\tau). \tag{5.1}$$

Assuming that the normalization of the factorizing operators is chosen according to equation (3.24) one can obtain the following determinant representation for the transfer matrix (5.1):

$$t_\rho(u,\tau) = (\mathcal{P}\tau^H)^{-N+1}\begin{vmatrix} \mathcal{Q}_1(u+\rho_1,\tau) & \mathcal{Q}_1(u+\rho_2,\tau) & \ldots & \mathcal{Q}_1(u+\rho_N,\tau)\\ \mathcal{Q}_2(u+\rho_1,\tau) & \mathcal{Q}_2(u+\rho_2,\tau) & \ldots & \mathcal{Q}_2(u+\rho_N,\tau)\\ \vdots & \vdots & \ddots & \vdots\\ \mathcal{Q}_N(u+\rho_1,\tau) & \mathcal{Q}_N(u+\rho_2,\tau) & \ldots & \mathcal{Q}_N(u+\rho_N,\tau)\end{vmatrix}. \tag{5.2}$$

[10] The choice $w \neq \sigma$ could be useful for the analysis of the homogeneous spin chains with a finite-dimensional quantum space.

The proof of (5.2) is based on the Berstein–Gel'fand–Gel'fand resolution of the finite-dimensional modules and will be given elsewhere [52]. Equation (5.2) gives rise to a variety of functional relations involving the Baxter $\mathcal{Q}$-operators and (in)finite-dimensional transfer matrices. The simplest of them are the so-called Wronskian relation and Baxter equation. One easily derives from equation (5.2) the Wronskian relation, which in $\boldsymbol{\lambda}$ notation, $t_\rho(u,\tau) \to t_{\boldsymbol{\lambda}}(u,\tau) = t_{\lambda_1\ldots\lambda_{N-1}}(u,\tau)$ reads

$$(\mathcal{P}\tau^H)^{N-1} t_{0\ldots0}(u,\tau) = \det|\mathcal{Q}_k(u+N-j,\tau)|_{k,j=1,\ldots,N}, \tag{5.3}$$

where the transfer matrix $t_{0\ldots0}(u,\tau)$ is proportional to the unit operator on quantum space.

Let us put

$$t_0(u,\tau) = t_N(u,\tau) = t_{0\cdots0}(u,\tau) \qquad \text{and} \qquad t_k(u,\tau) = t_{0\ldots-1_k\ldots0}(u,\tau). \tag{5.4}$$

That is the transfer matrix $t_k(u,\tau)$ is given by a trace of a monodromy matrix over a finite-dimensional auxiliary space which corresponds to the Young tableu with one column and k rows. Following the lines of [37] one can derive the self-consistency equation (Baxter equation) involving the Baxter $\mathcal{Q}$-operators and the finite-dimensional transfer matrices, $t_k(u,\tau)$. It takes the form of the Nth order difference equation

$$\sum_{k=0}^{N} (-1)^k t_k(u+k/N,\tau)\mathcal{Q}_j(u+N-k,\tau) = 0, \tag{5.5}$$

which, due to equation (4.8), can be considered as a fusion relation involving the finite- and infinite-dimensional transfer matrices of a special type. Equation (5.5) is a generalization of the standard $sl(2)\,T$–Q relation.

Thus the operators $\mathcal{Q}_k(u)$ possess the following properties:

- form a commutative family $[\mathcal{Q}_k(u), Q_j(v)] = 0$;
- commute with all transfer matrices $[\mathcal{Q}_k(u), \mathsf{T}_\rho(v)] = 0$;
- satisfy the Nth order difference equation (5.5) involving the finite-dimensional transfer matrices.

The operators with such properties are usually referred to as the Baxter $\mathcal{Q}$-operators, which justifies using this name in the previous sections.

Let us note that relations similar to (5.2), (5.3), (5.5) hold for the spin chains with the affine $U_q(\widehat{sl}(n))$ symmetry algebra [19, 30, 53] (see also [54, 55]).

It follows from equations (4.17), (5.2) that the Baxter operators encode the full information about the system. Provided that the eigenvalues of the Baxter operators are known one can restore eigenvalues of all transfer matrices. For low-rank symmetry models knowledge of the eigenvalues of the Baxter operators is sufficient to restore the wavefunction of the system. This can be done with the help of the separation of variables method developed by Sklyanin [4]. (Applications for the specific models can be found in [9, 20, 23, 56–59].)

The Hamiltonian of the system is defined as a logarithmic derivative of the transfer matrix with the same representation in the quantum and auxiliary space. It can be represented as the sum of pairwise Hamiltonians

$$\mathsf{H} = \frac{\mathrm{d}}{\mathrm{d}u}\log \mathsf{T}_\sigma(u,\tau)\Big|_{u=0} = \sum_{k=1}^{L} \tau^{H_k}\mathcal{H}_{kk+1}\tau^{-H_k}, \tag{5.6}$$

where

$$\mathcal{H}_{kk+1} = \mathcal{R}^{-1}_{kk+1}(0)\frac{\mathrm{d}}{\mathrm{d}u}\mathcal{R}_{kk+1}(u)\big|_{u=0}. \tag{5.7}$$

Using equations (3.42) and (3.22) one easily finds the following expression for the kernel of the pairwise Hamiltonian (5.7):

$$\mathcal{H}(z, w|\alpha\beta) = \log\left(\prod_{m=1}^{N}\left(\bar\beta_w w^{-1} z \bar\alpha_z^{-1}\right)_{mm}\right)\mathcal{I}^{\sigma}(z,\alpha)\mathcal{I}^{\sigma}(w,\beta). \tag{5.8}$$

Let us note that for the $sl(2)$ case the argument of the logarithmic function turns into the invariant ratio $(1+z\bar\beta)(1+w\bar\alpha)/((1+z\bar\alpha)(1+w\bar\beta))$. (Here $z = z_{21}, w = w_{21}$, etc.) The eigenvalues of the Hamiltonian (5.6) can be expressed in terms of the eigenvalues of the Baxter operators as follows:

$$\mathsf{E} = \sum_{k=1}^{L}\frac{\mathrm{d}}{\mathrm{d}u}\log\mathcal{Q}_k(u+\sigma_k)\bigg|_{u=0}. \tag{5.9}$$

For a generic representation of the $sl(N)$ algebra it is not possible to define an invariant scalar product and the Hamiltonian (5.6) is not Hermitian in general. Of special interest is a situation when the Verma modules possess an invariant submodule which admits an invariant scalar product. It could be a finite-dimensional subrepresentation which admits the $SU(N)$ invariant scalar product. There are also infinite-dimensional invariant subspaces which can be equipped with invariant scalar products. They can be identified with unitary representations of the noncompact group $SU(m, N-m)$. The spin chain with the $SU(2,2)$ symmetry group, for instance, is relevant for a description of the anomalous dimensions of a certain class of composite operators in quantum chromodynamics [60]. We hope that the approach developed here will be useful for an analysis of this type of models.

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Appendix A. The $SL(N,\mathbb{C})$ factorizing operator in the coherent state basis

We start the proof of equation (2.45) with a remark that the coherent state (2.43) can be represented in the form

$$\Delta^{\sigma\rho}(z, w|\alpha,\beta) = T^{\alpha}(\alpha^{-1})T^{\beta}(\beta^{-1})\cdot 1. \tag{A.1}$$

Let us apply the operator $\mathbb{R}^{(m)}$ to the function

$$\Phi(z,w) = \int D\alpha D\beta f(\alpha,\beta)\Delta^{\sigma\rho}(z,w|\alpha,\beta), \tag{A.2}$$

where $f(\alpha,\beta)$ is a smooth function with a finite support.

The evaluation of $\mathbb{R}^{(m)}\Phi$ is based on the identity

$$\begin{aligned}\mathbb{R}^{(m)}(\lambda)T^{\alpha}(\alpha^{-1})T^{\beta}(\beta^{-1}) &= \mathbb{W}_2[(w^{-1}z)_{N1}]^{\lambda}\mathbb{W}_1 T^{\alpha}(\alpha^{-1})T^{\beta}(\beta^{-1})\\ &= T^{\alpha_{m,\lambda}}(\alpha^{-1})T^{\beta_{m,-\lambda}}(\beta^{-1})\mathbb{W}_2[(w^{-1}\beta\alpha^{-1}z)_{N1}]^{\lambda}\mathbb{W}_1,\end{aligned} \tag{A.3}$$

where

$$\mathbb{W}_1 = \left(\overleftarrow{\prod_{j=m}^{N-1}}\mathbb{V}_j(\rho_{m,j+1})\right)\left(\overrightarrow{\prod_{i=1}^{m-1}}\mathbb{U}_i(\sigma_{im})\right), \tag{A.4}$$

$$\mathbb{W}_2 = \left(\prod_{i=1}^{\overleftarrow{m-1}} \mathbb{U}_i(\lambda - \sigma_{im})\right)\left(\prod_{j=m}^{\overrightarrow{N-1}} \mathbb{V}_j(\lambda - \rho_{m,j+1})\right). \tag{A.5}$$

To derive (A.3) one uses the following identities,

$$\mathbb{W}_1 T^{\alpha}(\alpha^{-1})T^{\beta}(\beta^{-1}) = T^{\alpha'}(\alpha^{-1})T^{\beta'}(\beta^{-1})\mathbb{W}_1, \tag{A.6}$$

$$[(w^{-1}z)_{N1}]^{\lambda} T^{\alpha'}(\alpha^{-1})T^{\beta'}(\beta^{-1}) = T^{\alpha''}(\alpha^{-1})T^{\beta''}(\beta^{-1})[(w^{-1}\beta\alpha^{-1}z)_{N1}]^{\lambda}, \tag{A.7}$$

$$\mathbb{W}_2 T^{\alpha''}(\alpha^{-1})T^{\beta''}(\beta^{-1}) = T^{\alpha_{m,\lambda}}(\alpha^{-1})T^{\beta_{m,-\lambda}}(\beta^{-1})\mathbb{W}_1, \tag{A.8}$$

where $\boldsymbol{\alpha}''(h) = h_{11}^{-\lambda}\boldsymbol{\alpha}'(h)$, $\boldsymbol{\beta}''(h) = h_{NN}^{\lambda}\boldsymbol{\beta}'(h)$ and

$$\boldsymbol{\alpha}'(h) = h_{11}^{\sigma_{1m}} \prod_{k=2}^{m} h_{kk}^{-\sigma_{k-1,k}} \boldsymbol{\alpha}(h), \qquad \boldsymbol{\beta}'(h) = h_{NN}^{-\rho_{mN}} \prod_{k=n}^{N-1} h_{kk}^{\rho_{k,k+1}} \boldsymbol{\beta}(h). \tag{A.9}$$

The identities (A.6) and (A.8) follow directly from the intertwining relation (2.37). To derive (A.7) it is sufficient to note

$$\left(w_{\beta}^{-1}\beta\alpha^{-1}z_{\alpha}\right)_{N1} = \left(d_{w,\beta}\beta_w w^{-1} z\alpha_z^{-1} d_{z,\alpha}^{-1}\right)_{N1} = (d_{w,\beta})_{NN}(w^{-1}z)_{N1}\left(d_{z,\alpha}^{-1}\right)_{11}, \tag{A.10}$$

where we recall that $\alpha \cdot z = z_{\alpha} d_{z,\alpha} \alpha_z$.

Let us now calculate $\mathbb{W}_2[(w^{-1}\beta\alpha^{-1}z)_{N1}]^{\lambda}\mathbb{W}_1$. To this end one can use the integral representation for the intertwining operators (2.35). Let us show that

$$\left(\prod_{j=m}^{\overrightarrow{N-1}} \mathbb{V}_j(\lambda - \rho_{m,j+1})\right)[(w^{-1}\beta\alpha^{-1}z)_{N1}]^{\lambda}\left(\prod_{j=m}^{\overleftarrow{N-1}} \mathbb{V}_j(\rho_{m,j+1})\right) = r_m(\lambda)[(w^{-1}\beta\alpha^{-1}z)_{m1}]^{\lambda}, \tag{A.11}$$

where

$$r_m(\lambda) = \prod_{j=m+1}^{N} \frac{A(\lambda - \rho_{mj})}{A(-\rho_{mj})} = \prod_{j=m+1}^{N} \frac{A(u_m - v_j)}{A(v_m - v_j)}, \tag{A.12}$$

and the parameters $u_k, v_k, \lambda, \sigma_k, \rho_k$ are related by (2.47). We represent the lhs of (A.11) in the form

$$\prod_{j=m+1}^{N} A(\rho_{mj})A(\rho_{jm} + \lambda) \int d^2\xi'_{j-1} \int d^2\xi_{j-1}[\xi_{j-1}]^{-(1+\lambda+\rho_{jm})}$$
$$\times [\xi'_{j-1} - \xi_{j-1}]^{-(1+\rho_{mj})}\left[\left(\left(w_{\xi}^{m}\right)^{-1}\beta\alpha^{-1}z\right)_{N1}\right]^{\lambda}, \tag{A.13}$$

where

$$w_{\xi}^{m} = w(1 - \xi_m e_{m+1m}) \cdots (1 - \xi_{N-1}e_{NN-1}). \tag{A.14}$$

It follows from (A.3) that the integrations in equation (A.13) have to be carried out in the following order $\xi_m, \ldots, \xi_{N-1}, \xi'_{N-1}, \ldots, \xi'_m$. Since the integrals converge, we rearrange them and carry out the integration over (ξ_{N-1}, ξ'_{N-1}), then over (ξ_{N-2}, ξ'_{N-2}), and so on till (ξ_m, ξ'_m). Each integration is reduced to the standard integral

$$\int d^2\xi' \int d^2\xi [\xi]^{-1-\lambda+\rho}[\xi' - \xi]^{-1-\rho}[\xi - x_0]^{\lambda} = \frac{1}{A(\rho)A(-\rho)}. \tag{A.15}$$

(We recall here that $[a]^{\lambda} = a^{\lambda}(a^*)^{\bar{\lambda}}$.) Then collecting all factors and taking into account that $(e_{NN-1}\cdots e_{m+1m}w^{-1}\beta\alpha^{-1}z)_{N1} = (w^{-1}\beta\alpha^{-1}z)_{m1}$ one gets equation (A.11).

Quite similarly one obtains

$$\left(\prod_{i=1}^{\overleftarrow{m-1}}\mathbb{U}_i(\lambda-\sigma_{im})\right)[(w^{-1}\beta\alpha^{-1}z)_{m1}]^{\lambda}\left(\prod_{i=1}^{\overrightarrow{m-1}}\mathbb{U}_i(\sigma_{im})\right)=p_m(\lambda)[(w^{-1}\beta\alpha^{-1}z)_{mm}]^{\lambda}, \qquad (A.16)$$

where

$$p_m(\lambda)=\prod_{k=1}^{m-1}(-1)^{\lambda-\bar\lambda}\frac{A(\lambda-\sigma_{km})}{A(-\sigma_{km})}=\prod_{k=1}^{m-1}(-1)^{\lambda-\bar\lambda}\frac{A(u_k-v_m)}{A(u_k-u_m)}. \qquad (A.17)$$

Using these results one obtains from (A.3)

$$\begin{aligned}\mathbb{R}^{(m)}(\lambda)\Delta^{\sigma\rho}(z,w|\alpha,\beta)&=f^{(m)}_{\sigma\rho}(\lambda)T^{\alpha_{m,\lambda}}(\alpha^{-1})T^{\beta_{m,-\lambda}}(\beta^{-1})[(w^{-1}\beta\alpha^{-1}z)_{mm}]^{\lambda}\\&=f^{(m)}_{\sigma\rho}(\lambda)\left[\left(\beta_w w^{-1}z\alpha_z^{-1}\right)_{mm}\right]^{\lambda}\Delta^{\sigma\rho}(z,w|\alpha,\beta). \qquad (A.18)\end{aligned}$$

Appendix B. Proof of the relations (2.38a) and (2.38*b*)

In this appendix we give the proof of the relations (2.40) and (2.38*b*) for the factorizing operators $\mathbb{R}^{(m)}$. We start from equations (2.40) and represent the operator $D_{k+1,k}$ in the form (see equation (2.11))

$$D_{k+1,k}=-\sum_{in}(z^{-1})_{ki}z_{n,k+1}E_{ni}. \qquad (B.1)$$

Since the relations (2.40) hold for the $SL(N,\mathbb{C})$ factorizing operators one can derive the following equation for the kernel: $\mathcal{K}^{(m)}_{\lambda,\sigma,\rho}(z,w|\alpha,\beta)$,

$$D^{(z)}_{k+1,k}\mathcal{K}^{(m)}_{\lambda,\sigma,\rho}(z,w|\alpha,\beta)=\sum_{in}(z^{-1})_{ki}z_{n,k+1}\widetilde{E}^{(\alpha)}_{ni}\mathcal{K}^{(m)}_{\lambda,\sigma,\rho}(z,w|\alpha,\beta), \qquad (B.2)$$

where $k>m+1$ and we have used the commutation relations (2.39) for the $SL(N,\mathbb{C})$ operators. Taking into account that the kernel of the $sl(N)$ factorizing operator $\mathbb{R}^{(m)}(\lambda)$ is given by the function $\mathcal{K}^{(m)}_{\lambda,\sigma,\rho}(z,w|\bar\alpha,\bar\beta)$ and making use of equations (3.14) one can easily check that equation (B.2) gives rise to the following equation:

$$D^{(z)}_{k+1,k}\mathbb{R}^{(m)}(\lambda)=\mathbb{R}^{(m)}(\lambda)D^{(z)}_{k+1,k} \qquad (B.3)$$

for the $sl(N)$ factorizing operator $\mathbb{R}^{(m)}(\lambda)$.

The relation (2.38*b*) is equivalent to the following equation for the kernels:

$$\mathcal{R}^{(m)}_{\lambda+\mu,\sigma\rho}(\eta,\xi|\bar\alpha,\bar\beta)=\Omega_{\sigma'\rho'}\left(\mathcal{R}^{(m)}_{\mu,\sigma'\rho'}(\eta,\xi|\bar z,\bar w)\mathcal{R}^{(m)}_{\lambda,\sigma\rho}(z,w|\bar\alpha,\bar\beta)\right). \qquad (B.4)$$

This equation is a consequence of the commutation relations (2.39).

Let us rewrite the kernel $\mathcal{R}^{(m)}_{\lambda,\sigma\rho}(z,w|\bar\alpha,\bar\beta)$ as follows (for brevity we will omit the normalization factor A_m):

$$\left(\bar\beta_w w^{-1}z\bar\alpha_z^{-1}\right)^{\lambda}_{mm}\mathcal{I}^{\sigma}(z,\alpha)\mathcal{I}^{\rho}(w,\beta)=\left(w_\beta^{-1}\bar\beta^{-1}\bar\alpha^{-1}z_\alpha\right)^{\lambda}_{mm}\mathcal{I}^{\sigma'}(z,\alpha)\mathcal{I}^{\rho'}(w,\beta). \qquad (B.5)$$

The key point is that the factor $X_m=\left(w_\beta^{-1}\bar\beta^{-1}\bar\alpha^{-1}z_\alpha\right)_{mm}$ depends only on the variables z^{-1}_{jk}, with $j>m$ and on the variables w_{kj} with $j<m$. It can be shown as follows (we consider only the z case): X_m depends on the variables $(z_\alpha)_{km}$ with $k=m+1,\ldots,N$, which, due to the triangularity of the matrix z_α, are given by linear combinations of the elements $(z_\alpha^{-1})_{kp}$ with $p\geqslant m$. The matrix z_α^{-1} is determined by the equation

$$z^{-1}\bar\alpha^{-1}=\bar\alpha_z^{-1}d^{-1}_{z,\alpha}z_\alpha^{-1}, \qquad (B.6)$$

which follows from equation (2.44). It is straightforward to see from the above equation that the matrix elements $(z_\alpha^{-1})_{kn}$ depend on the elements z_{jn}^{-1} with $j \geqslant k$ only. Therefore, taking into account the commutation relations of equation (2.39) one can transform (B.4) to

$$\begin{aligned}\Omega_{\sigma'\rho'}\big(\mathcal{R}^{(m)}_{\mu,\sigma'\rho'}(\eta,\xi|\bar{z},\bar{w})\big(\xi_\beta^{-1}\bar{\beta}^{-1}\bar{\alpha}^{-1}\eta_\alpha\big)^\lambda_{mm}\mathcal{I}^{\sigma'}(z,\alpha)\mathcal{I}^{\rho'}(w,\beta)\big)\\ = \big(\xi_\beta^{-1}\bar{\beta}^{-1}\bar{\alpha}^{-1}\eta_\alpha\big)^\lambda_{mm}\Omega_{\sigma'\rho'}\big(\mathcal{R}^{(m)}_{\mu,\sigma'\rho'}(\eta,\xi|\bar{z},\bar{w})\mathcal{I}^{\sigma'}(z,\alpha)\mathcal{I}^{\rho'}(w,\beta)\big)\\ = \big(\xi_\beta^{-1}\bar{\beta}^{-1}\bar{\alpha}^{-1}\eta_\alpha\big)^\lambda_{mm}\mathcal{R}^{(m)}_{\mu,\sigma'\rho'}(\eta,\xi|\bar{\alpha},\bar{\beta}) = \mathcal{R}^{(m)}_{\mu+\lambda,\sigma\rho}(\eta,\xi|\bar{\alpha},\bar{\beta}),\end{aligned} \tag{B.7}$$

where in the last step we use the transformation (B.5).

Appendix C. Proof of the estimate (4.16) for matrix elements of factorizing operators

Here we derive the estimate (4.16) for the matrix element of the factorizing operator $\mathcal{R}^{(k)}_{10}(\lambda)$. Let $e_n(z)$ and $e_k(w)$ be the basis vectors (3.5) in the quantum and auxiliary spaces, respectively. Defining

$$E_n(z) = \prod_{i>k}\frac{z^{n_{ik}}}{n_{ik}!} \tag{C.1}$$

one can represent $e_n(z)$ as

$$e_n(z) = \Omega_{kn}(\boldsymbol{\sigma})E_k(\partial_{\bar{\alpha}})\mathcal{I}^{\sigma}(z,\alpha)|_{\alpha=1(\alpha_{ij}=0,i>j)}, \tag{C.2}$$

where $\Omega_{kn}(\boldsymbol{\sigma}) \equiv \Omega_{\boldsymbol{\sigma}}(\bar{e}_k, e_n)$ and the sum over repeating (multi)indices is implied. The matrix element $[\mathcal{R}^{(k)}_{10}(\lambda)]^{m'n'}_{mn}$ can be expressed as follows:

$$\big[\mathcal{R}^{(k)}_{10}(\lambda)\big]^{m'n'}_{mn} = \Omega_{ni}(\boldsymbol{\rho})\Omega_{mj}(\boldsymbol{\sigma})E_{m'}(\partial_z)E_{n'}(\partial_w)E_j(\partial_{\bar{\alpha}})E_i(\partial_{\bar{\beta}})\mathcal{R}^{(m)}_{\lambda,\sigma\rho}(w,z|\bar{\alpha},\bar{\beta})\big|_{z=w=\alpha=\beta=1}. \tag{C.3}$$

We are interested in the behavior of this matrix element at $n_{jk}\to\infty$, $j=k+1,\dots,N$, all other indices m_{ji}, m'_{ji} and $n_{ji}, n'_{ji}, i\neq k$ being fixed. It is clear that the multi-index j on the rhs of (C.3) varies in finite limits. Using equation (3.22) for $\mathcal{R}^{(m)}_{\lambda,\sigma\rho}$ and carrying out differentiation with respect to $\bar{\alpha}$ and z one finds that the matrix element (C.3) is given by a sum of terms which have the form

$$\Omega_{ni}(\boldsymbol{\rho})E_{n'}(\partial_w)E_i(\partial_{\bar{\beta}})(\bar{\beta}w)^{\lambda-K}_{kk}P(w,\bar{\beta})|_{w=\beta=1}, \tag{C.4}$$

where K is some constant. The polynomial $P(w,\bar{\beta})$ has a finite degree and does not contain large factors n_{jk}. The factor $(\bar{\beta}w)_{kk} = 1+\sum_p\bar{\beta}_{pk}w_{pk}$ depends only on the 'large' variables $w_{pk}, \bar{\beta}_{pk}$. After differentiation with respect to all other variables one gets for (C.4)

$$\Omega_{ni}(\boldsymbol{\rho})\prod_{p=k}^{N}\frac{1}{n'_{pk}!}\frac{1}{i_{pk}!}\left(\frac{\partial}{\partial w_{pk}}\right)^{n'_{pk}}\left(\frac{\partial}{\partial\bar{\beta}_{pk}}\right)^{i_{pk}}(\bar{\beta}w)^{\lambda-K}_{kk}\widetilde{P}(w_{pk},\bar{\beta}_{pk})\bigg|_{w_{pk}=\beta_{pk}=0}, \tag{C.5}$$

where again the polynomial $\widetilde{P}$ does not contain large factors n_{jk}. Finally, one gets that (C.5) is given by the (finite) sum of the terms

$$\Omega_{ni}(\boldsymbol{\rho})\frac{\Gamma\big(-\lambda+K+\sum_{j=k+1}^{N}(n'_{jk}-s_j)\big)}{(n'_{k+1k}-s_{k+1})!\cdots(n'_{Nk}-s_N)!}\times R(n'_{k+1k},\dots,n'_{Nk}), \tag{C.6}$$

where s_j are some constants, R is a polynomial and the difference of indices $i_{jk}-n'_{jk}$ is finite at $n'_{jk}\to\infty$, $j=k+1,\dots,N$.

Let us estimate the coefficient $\Omega_{ni}(\boldsymbol{\rho})$ when all indices except n_{jk} and i_{jk}, $j > k$ are small. To this end we will use the integral representation (3.15),

$$\Omega_{ni}(\boldsymbol{\rho}) = c_N(\boldsymbol{\rho}) \int Dz\, \mu_N(z)\overline{e_n(z)}e_i(z), \tag{C.7}$$

with $\mu_N(z) = \prod_{j=1}^{N-1} \Delta_j^{-\lambda_j-1}(z^\dagger z)$, $\lambda_j = \rho_j - \rho_{j+1}$. We recall that by definition the integral (C.7) for arbitrary $\{\lambda_j\}$ is understood as an analytic continuation from the region of λ' s where it converges. To stress that the matrix element $\Omega_{ni}(\boldsymbol{\rho})$ is considered in the special 'kinematic' $(n_{jk}, j_{jk} \to \infty)$ we put more labels on it, $\Omega_{ni}(\boldsymbol{\rho}) \to \Omega_{ni}^{N,k}(\boldsymbol{\lambda})$, and switch from $\boldsymbol{\rho}$ to $\boldsymbol{\lambda}$.

To get the necessary estimate we proceed as follows: first we show that the coefficient $\Omega_{ni}^{N,k}(\boldsymbol{\lambda})$ for $k > 1$ can be represented as the sum of integrals (C.7) for $sl(N-1)$ case

$$\Omega_{ni}^{N,k}(\boldsymbol{\lambda}) = \sum_q c_q \Omega_{n_q,i_q}^{N-1,k-1}(\boldsymbol{\lambda}_q). \tag{C.8}$$

It is important that the sum in (C.8) over q goes in a finite range and the coefficients c_q and parameters $(\lambda_q)_j = \lambda_j + \delta_j$ do not depend on n_{jk}, i_{jk}. Continuing this procedure one can represent $\Omega_{ni}^{N,k}(\boldsymbol{\lambda})$ as the sum of the elements $\Omega_{n_qi_q}^{M,1}(\boldsymbol{\lambda}_q)$, where $M = N-k+1$, thus reducing the problem to an evaluation of the matrix elements $\Omega_{n_qi_q}^{M,1}(\boldsymbol{\lambda}_q)$.

To prove (C.8) let us make the change of variables $z = w^{-1}$ and carry out the integrations over variables $w_{21}, \ldots, w_{N1}$. Let us represent matrix w as

$$w = \begin{pmatrix} 1 & \vec{0} \\ \vec{a} & b \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ a_1 & 1 & 0 & \ldots & 0 \\ a_2 & b_{21} & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{N-1} & b_{N-11} & \ldots & b_{N-1,N-2} & 1 \end{pmatrix} \tag{C.9}$$

and examine dependence of Δ_p on the elements $a_1, \ldots, a_{N-1}$. For $N \times N$ matrix M we define $\widetilde{\Delta}_p(M) = \det M_p$ and M_p is $(N-p)\times(N-p)$ matrix with elements M_{ij}, $i, j = p+1, \ldots, N$. Noting that $\Delta_p(z^\dagger z) = \widetilde{\Delta}_p(ww^\dagger) = \det|(bb^\dagger)_{ij} + a_i\bar{a}_j|_{i,j=p,\ldots,N-1}$ one concludes that

$$\begin{aligned} &\Delta_1 = \Delta_1(a_1, \ldots, a_{N-1}), \qquad \Delta_2 = \Delta_2(a_2, \ldots, a_{N-1}), \\ &\ldots \qquad \Delta_{N-1} = \Delta_{N-1}(a_{N-1}). \end{aligned} \tag{C.10}$$

Let $B = bb^\dagger$ and $A_p = (a_p, \ldots, a_{N-1})$. Taking into account that

$$\widetilde{\Delta}_p(ww^\dagger) = \det\left(B_{p-1} + A_p \otimes A_p^\dagger\right) = \widetilde{\Delta}_{p-1}(bb^\dagger)\cdot\left(1 + A_p^\dagger B_{p-1}^{-1} A_p\right) \tag{C.11}$$

one gets for the measure

$$\begin{aligned} \mu_N(w) &= \prod_{p=1}^{N-1} \widetilde{\Delta}_p^{-\lambda_p-1}(ww^\dagger) = \prod_{p=1}^{N-1} \widetilde{\Delta}_{p-1}^{-\lambda_p-1}(bb^\dagger)\left(1 + A_p^\dagger B_{p-1}^{-1} A_p\right)^{-\lambda_p-1} \\ &= \mu_{N-1}(b) \prod_{p=1}^{N-1} \left(1 + A_p^\dagger B_{p-1}^{-1} A_p\right)^{-\lambda_p-1}, \end{aligned} \tag{C.12}$$

where $\mu_{N-1}(b) = \prod_{p=1}^{N-2} \widetilde{\Delta}_p^{-\lambda_{p+1}-1}(bb^\dagger)$ $(\widetilde{\Delta}_0(bb^\dagger) = 1)$. Now one can consequently carry out the integration over $a_1, a_2, \ldots, a_{N-1}$. Indeed, let us consider integral

$$\int \mathrm{d}^2 a_1\, a_1^m \bar{a}_1^{\bar{m}} \left(1 + A_1^\dagger B^{-1} A_1\right)^{-\lambda_1-1}. \tag{C.13}$$

Representing $B^{-1}_{ij} = m_{ij}/\det B$, where $m_{ij} = (-1)^{i+j} M_{ij}$, M_{ij} being a minor of the matrix B, one derives

$$\begin{aligned}\left(1 + A_1^\dagger B^{-1} A_1\right) &= \left(1 + \frac{1}{\det B}\left[|a_1|^2 m_{11} + a_1^\dagger m_{1j} a_j + a_i^\dagger m_{i1} a_1 + a_i^\dagger m_{ij} a_j\right]\right)\\ &= \left(1 + \frac{m_{11}}{\det B}\left[|a_1 + m_{1j}/m_{11} a_j|^2 + \frac{1}{m_{11}^2} a_i^\dagger \{m_{11} m_{ij} - m_{i1} m_{1j}\} a_j\right]\right)\\ &= \left(1 + \frac{m_{11}}{\det B} |a_1 + m_{1j}/m_{11} a_j|^2 + a_i^\dagger \left(B_1^{-1}\right)_{ij} a_j\right).\end{aligned} \tag{C.14}$$

Here the summation over repeated indices ($i, j = 2, \ldots, N-1$) is implied and we make use of the identity

$$\{m_{11} m_{ij} - m_{i1} m_{1j}\} = m_{11} \det B \cdot \left(B_1^{-1}\right)_{ij}. \tag{C.15}$$

After shifting and rescaling the integration variable one gets for (C.13)

$$\sum_{q=0}^{\min(m,\bar m)} c_q(\lambda_1) \left(\frac{\det B}{m_{11}}\right)^{q+1} \left(-\frac{m_{1j} a_j}{\det B}\right)^{m-q} \left(-\frac{\bar m_{1j} \bar a_j}{\det B}\right)^{\bar m - q} \left(1 + A_2^\dagger B_1^{-1} A_2\right)^{-\lambda_1 + q}, \tag{C.16}$$

where $c_q(\lambda) = C_q^m C_q^{\bar m} q! \Gamma(\lambda - q)/\Gamma(\lambda + 1)$ and $m_{11} = \det B_1 = \widetilde{\Delta}_1 (bb)^\dagger$. Evidently, integrating over $a_2, a_3, \ldots$ one again encounters the integrals of the type (C.13), hence the final result of integrations can be cast into the form (C.8). This calculation also shows that the integral (C.7) is a meromorphic function of $\lambda_1, \ldots, \lambda_{N-1}$.

In order to calculate the element $\Omega^{N,1}_{ni}(\lambda_q)$ one can repeatedly integrate over the variables z_{Nj}, $j > 1$ in the last row, then over z_{N-1j}, $j > 1$ in the row $N-1$ and so on. Representing the matrix z in the form

$$z = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0\\ b_{21} & 1 & 0 & \ldots & 0\\ \vdots & \vdots & \ddots & & \vdots\\ b_{N-11} & b_{N-12} & \ldots & 1 & 0\\ a_1 & a_2 & \ldots & a_{N-1} & 1\end{pmatrix}, \tag{C.17}$$

$(z^\dagger z)_{ij} = (b^\dagger b)_{ij} + \bar a_i a_j$, for $i, j = 1, \ldots, N-1$, one easily finds that $\Delta_k(z)$ depends only on the variables $a_1, \ldots, a_k$. The integration over $a_{N-1}, a_{N-2}, \ldots, a_2$ goes along the same lines as before. For instance,

$$\prod_{k=2}^{N-1} \int d^2 a_k \prod_{k=2}^{N-1} \Delta_k^{-\lambda_k - 1}(z^\dagger z) = \gamma_{N-1} \left[\prod_{k=2}^{N-2} \Delta_k^{-\lambda_k - 1}(b^\dagger b)\right] R(z_{21}, \ldots, z_{N1})$$

where $\gamma_{N-1} = \pi^{N-2}(\lambda_{N-1}(\lambda_{N-1} + \lambda_{N-2}) \cdots (\lambda_{N-1} + \cdots + \lambda_2))^{-1}$ and

$$R(z_{21}, \ldots, z_{N1}) = \frac{(1 + |z_{21}|^2 + \cdots + |z_{N-11}|^2)^{\lambda_2 + \cdots + \lambda_{N-1} - 1}}{(1 + |z_{21}|^2 + \cdots + |z_{N1}|^2)^{\lambda_2 + \cdots + \lambda_{N-1}}}.$$

Therefore, one gets

$$\begin{aligned}&\left(\prod_{1<j<k\leqslant N-1} \int d^2 z_{kj}\right) \mu_N(z) \overline{e_n(z)} e_i(z)\\ &\quad = \sum_q c(\lambda, q) \prod_{p=2}^{N} z_{p1}^{m_p} \bar z_{p1}^{l_p} \prod_{n=2}^{N-1} \left(1 + \sum_{k=2}^{n} |z_{k1}|^2\right)^{\mu_n - 1} \left(1 + \sum_{k=2}^{N} |z_{k1}|^2\right)^{-\Lambda_N - 1},\end{aligned} \tag{C.18}$$

where $m_p = i_{p1} + r_p(q)$, $l_p = n_{p1} + s_p(q)$, $\mu_p = \lambda_p + \delta_p(q)$ and $\Lambda_N = \mu_1 + \cdots + \mu_{N-1}$. The parameters $c(\lambda, q)$, $r_p(q)$, $s_p(q)$, $\delta_p(q)$ do not depend on $n_{k1}, i_{k1}, k = 2, \ldots, N$ and the over multi-index q goes in the finite limits. Finally, after integration over z_{p1}, $p = 2, \ldots, N$ one gets for each term in the sum

$$\pi^{N-1} c(\lambda, q) \prod_{p=2}^{N} (-1)^{m_p+1} \delta_{m_p, l_p} m_p! \prod_{k=3}^{N} \frac{\Gamma(1 - \Lambda_{k-1} + M_k)}{\Gamma(1 - \Lambda_k + M_k)} \frac{\Gamma(1 - \Lambda_N)}{\Gamma(1 - \Lambda_2 + M_2)}, \tag{C.19}$$

where $\Lambda_k = \mu_1 + \cdots + \mu_{k-1}$ and $M_k = m_k + \cdots + m_N$. Together with equation (C.6) this results in the estimate (4.16).

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